Ambient noise cross correlation in free space: Theoretical approach

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It has been experimentally demonstrated that the Green's function between two points could be recovered using the cross-correlation function of the ambient noise measured at these two points. This paper investigates the theory behind this result in the simple case of a homogeneous medium with attenuation. © *2005 Acoustical Society of America.* [DOI: 10.1121/1.1830673]

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I. INTRODUCTION

The goal of this work is to theoretically investigate the following problem: can we retrieve the time-domain Green's function (TDGF) between two points by performing a cross correlation of the ambient noise field received on those two points? Experimental demonstration of this process has been performed in ultrasonic,[1] underwater acoustics,[2] or geophysics.[3,4] In most cases, only an estimate of the Green's function was retrieved, with quality strongly dependent on the medium complexity as well as the spatio-temporal distribution of the ambient noise sources. For example, it has been shown that an amplitude-shaded TDGF could be obtained with surface ambient noise in underwater acoustics, the shading being due to the surface location of noise sources in the ocean.[2,5] Similarly, Lobkis and Weaver[6] have shown the emergence of the exact TDGF from the diffuse field due to thermal fluctuations in a reverberant ultrasonic cavity. Finally, earlier works by Rickett and Claerbout conjectured that this process could also be used to retrieve the sound-speed structure of the upper crust of the earth in geophysics, the experimental demonstration being done in helioseismology from data describing the random vibration of the sun's surface.[3] Because noise sources are difficult to work with, some related results have also been obtained with noise-like events, where signals recorded from randomly distributed sources were used to perform the cross correlation.[7,8] For example, in Ref. 7, the cross correlation of the coda waves from a distribution of seismic events provided the Rayleigh wave between two seismometers.

From a theoretical point of view, earlier works have investigated the problem of spatial correlation with noise fields[9,1] or with wave fields obtained from a distribution of random sources.[10,11] In the case of a 3D free-space medium with a spatially uniform noise source distribution, the field at each receiver can be decomposed as a superposition of uncorrelated plane waves from various directions. It has been established[9] that the normalized cross-spectral density $C_{1,2}(\omega)$ at frequency ω between two receivers 1 and 2 separated by a distance r is $C_{1,2}(\omega)=[\sin(\omega r/c)]/(\omega r/c)$. In the time domain, the normalized correlation function is

$$C_{1,2}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_{1,2}(\omega)\exp(i\omega t)\,d\omega,$$

which can be written as

$$C_{1,2}(t)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\exp[i\omega(t+r/c)]}{i\omega r/c}\,d\omega-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\exp[i\omega(t-r/c)]}{i\omega r/c}\,d\omega. \tag{1}$$

The time derivative of the correlation function is then

$$\frac{d}{dt}C_{1,2}(t)=\frac{1}{4\pi r/c}[\delta(t+r/c)-\delta(t-r/c)]. \tag{2}$$

The two terms in Eq. (2) correspond to the backward and forward Green's function between the receivers, which demonstrates the connection between the correlation function and the Green's function. However, the drawback of this elegant result is to start from a *normalized* correlation function, normalization that is required because the overall spatial contribution from noise sources in a lossless infinite medium is infinite. Experimentally though, ambient noise signals are always finite, as is the noise correlation function. The contradiction comes from the fact that the theory is developed in lossless environments while experiments are always performed in the presence of attenuation. Thus, normalization acts as a subterfuge for avoiding inclusion of the required attenuation in the theory. The goal of our work is to show how the result in Eq. (2) could be derived rigorously without the need of normalization when attenuation is present in the medium.

In this paper, we work directly with noise in the time domain. Thus, the mathematical developments in this paper start from an infinite-bandwidth formulation of the Green's function. To be as general as possible, we deal here with two receivers simultaneously recording ambient noise in a 3D homogeneous medium. The choice of free-space propagation has been made in regards to the complex developments

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needed to achieve the demonstration in a waveguide.[5] The incident field on the two receivers comes from a homogeneous spatial-temporal distribution of uncorrelated broadband noise sources. This model is reasonable in our case because we evaluate the ensemble average of the noise correlation function. It also presents the advantage of considerably simplifying the mathematical developments performed in Secs. II and III.

This paper is structured as follows. In Sec. II, we examine the case of a homogeneous space without attenuation. We use a geometrical interpretation to investigate the relationship between the noise correlation function and the Green's function. In Sec. III, we extend the results to a medium with attenuation. Section IV is a discussion that links this theoretical approach with earlier experimental works.

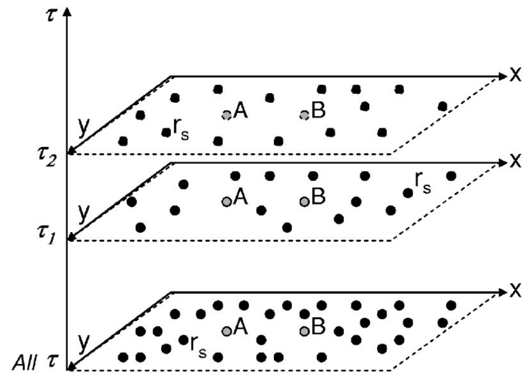

FIG. 1. Representation of the receivers in A and B and the noise sources (in $\mathbf{r}_s$) in the xy plane. Each plane corresponds to a new distribution of noise sources at time τ. The integral over τ in Eq. (6) could be done by accumulating all noise sources over time as shown in the lower plane.

II. FREE SPACE WITHOUT ATTENUATION

In a 3D homogeneous medium without attenuation, the Green's function between points A (in $\mathbf{r}_1$) and B (in $\mathbf{r}_2$) is

$$G(\mathbf{r}_2,t;\mathbf{r}_1,0)=\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}\,\delta\!\left(t-\frac{|\mathbf{r}_2-\mathbf{r}_1|}{c}\right), \tag{3}$$

where c is the constant sound speed in the medium. Assuming a random spatial-temporal distribution of noise sources amplitudes $S(\mathbf{r}_s,t_s)$, the total field received in A is

$$\begin{aligned}P(\mathbf{r}_1;t)&=\int_{-\infty}^{\infty}d\mathbf{r}_s\int_{-\infty}^{t}dt_s S(\mathbf{r}_s,t_s)G(\mathbf{r}_1,t;\mathbf{r}_s,t_s)\\&=\int_{-\infty}^{\infty}\frac{d\mathbf{r}_s}{|\mathbf{r}_1-\mathbf{r}_s|}S\!\left(\mathbf{r}_s,t-\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c}\right).\end{aligned} \tag{4}$$

Here, the causality requires that the noise sources in $(\mathbf{r}_s;t_s)$ that contribute to the pressure field in A at a given time t satisfy the condition $t=t_s+|\mathbf{r}_1-\mathbf{r}_s|/c$. Then, the cross correlation of the two signals recorded in A and B is defined as

$$C(\mathbf{r}_1,\mathbf{r}_2;t)=C_{1,2}(t)=\int_{-\infty}^{+\infty}P(\mathbf{r}_1;\tau)P(\mathbf{r}_2;t+\tau)d\tau, \tag{5}$$

which leads to

$$\begin{aligned}C_{1,2}(t)=&\int_{-\infty}^{+\infty}d\mathbf{r}_s\int_{-\infty}^{+\infty}d\mathbf{r}_{s'}\int_{-\infty}^{+\infty}d\tau\frac{1}{|\mathbf{r}_1-\mathbf{r}_s||\mathbf{r}_2-\mathbf{r}_{s'}|}\\&\times S\!\left(\mathbf{r}_s,\tau-\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c}\right)S\!\left(\mathbf{r}_{s'},\tau+t-\frac{|\mathbf{r}_2-\mathbf{r}_{s'}|}{c}\right).\end{aligned} \tag{6}$$

$C_{1,2}(t)$ corresponds to one realization of the ambient noise cross-correlation function. To evaluate the average noise correlation function over an ensemble of realization, we use the fact that noise sources are spatially and temporally uncorrelated

$$\langle S(\mathbf{r}_s,t_s)S(\mathbf{r}_{s'},t_{s'})\rangle=Q^2\delta(t_s-t_{s'})\,\delta(\mathbf{r}_s-\mathbf{r}_{s'}), \tag{7}$$

where the notation $\langle X\rangle$ corresponds to the ensemble average of X. Q^2 represents the acoustic power of the noise sources and is taken constant over time and space. Then, it follows, after integration over $d\mathbf{r}_{s'}$

$$\begin{aligned}\langle C_{1,2}(t)\rangle=&Q^2\int_{-\infty}^{+\infty}d\mathbf{r}_s\int_{-\infty}^{+\infty}d\tau\frac{1}{|\mathbf{r}_2-\mathbf{r}_s||\mathbf{r}_1-\mathbf{r}_s|}\\&\times\delta\!\left(t+\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c}-\frac{|\mathbf{r}_2-\mathbf{r}_s|}{c}\right).\end{aligned} \tag{8}$$

Note that, as written in Eq. (5), τ corresponds to the running time of the signals received in A and B. However, the cross-correlation function $C_{1,2}(t)$ is a comparison between those two signals, meaning that $C_{1,2}(t)$ will extract the relative propagation times between the noise source in $\mathbf{r}_s$, and the receivers in $\mathbf{r}_1$ and $\mathbf{r}_2$. As a consequence, the integration over τ corresponds just to an accumulation of noise sources over time. As shown in Fig. 1, all those noise events could be excited at one time only without changing the final result in the average correlation function. Actually, the integration over τ in Eq. (8) leads to the usual divergence of a correlation function over an infinite time. However, in a practical experimental case, the pressure fields in A and B first have to be recorded over a finite interval time T before the cross correlation is performed. Assuming that the random noise sources have a creation rate ν ($\mathrm{m}^{-3}\,\mathrm{s}^{-1}$) per unit time per volume, the integral over τ is changed into the product $T\nu$. The accumulation of noise events described in Fig. 1 corresponds then to an increase of ν over a smaller interval time T such that $T\nu$ remains constant. Finally, using this approach, Eq. (8) becomes

$$\begin{aligned}\langle C_{1,2}(t)\rangle=&Q^2T\nu\int_{-\infty}^{+\infty}d\mathbf{r}_s\frac{1}{|\mathbf{r}_2-\mathbf{r}_s||\mathbf{r}_1-\mathbf{r}_s|}\\&\times\delta\!\left(t+\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c}-\frac{|\mathbf{r}_2-\mathbf{r}_s|}{c}\right).\end{aligned} \tag{9}$$

Equation (9) shows that the noise correlation function in free space reduces to the calculation of a spatial integral over the noise source's locations. In the following, we show that a geometrical argument allows us to obtain an analytical solution for $\langle C_{1,2}(t)\rangle$. We first define a Cartesian coordinate system for the 3D space in which A is $(a,0,0)$, B is $(-a,0,0)$, and a point $\mathbf{r}_s$ is (x,y,z). The argument of the delta function in Eq. (9) gives a contribution to the correlation function at time t if $\mathbf{r}_s$ is such that $|\mathbf{r}_2-\mathbf{r}_s|-|\mathbf{r}_1-\mathbf{r}_s|$

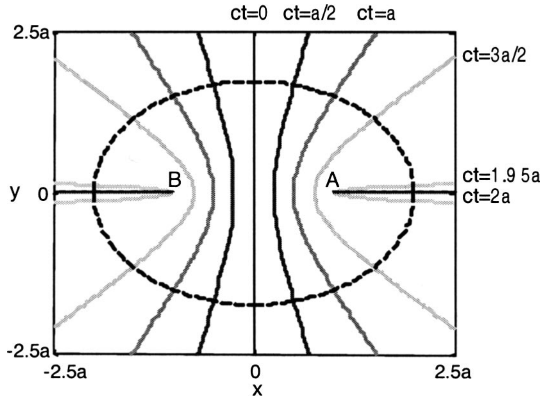

FIG. 2. Representation in the xy plane of the hyperbola that contributes to a given time t in the noise correlation function. On each hyperbola, the noise sources satisfy $|\mathbf{r}_2-\mathbf{r}_s|-|\mathbf{r}_1-\mathbf{r}_s|=ct$. The line $x=0$ corresponds to $ct=0$. The line $y=0$ corresponds to $ct=2a$ for $x\geqslant a$, and $ct=-2a$ for $x\leqslant -a$. The ellipse (dashed line) represents the noise sources for which $|\mathbf{r}_2-\mathbf{r}_s|+|\mathbf{r}_1-\mathbf{r}_s|=ct$, for $ct=4a$. The receivers A and B are at $(a,0)$ and $(-a,0)$.

$=ct$. For a time t satisfying the condition $-2a\leqslant ct\leqslant 2a$, the noise sources must lie on a hyperboloid defined by the equation (Fig. 2)

$$y^2+z^2-x^2\left(\frac{4a^2}{c^2t^2}-1\right)=a^2\left(1-\frac{c^2t^2}{4a^2}\right),\quad \text{if } ct\neq 0, \tag{10}$$

and $x=0$, if $ct=0$.

The hyperbola in Fig. 2 correspond to the noise sources in a 2D section at constant depth z of the 3D space that will contribute to $C_{1,2}(t)$ at a given time t. We will see in Sec. III the importance of the ellipse in Fig. 2 that is made of the noise sources in $\mathbf{r}_s$ such that $|\mathbf{r}_2-\mathbf{r}_s|+|\mathbf{r}_1-\mathbf{r}_s|=ct$. Note that these conical shapes are invariant by rotation around the axis of the two receivers (x axis in Fig. 2). This means that, in 3D, the hyperbola and the ellipse turn out to be a hyperboloid and an ellipsoid whose symmetry axis is the lines between the two receivers. As can be seen in Fig. 2, every noise source in space belongs to a unique hyperbola. Each hyperbola is parametrized by ct, with the condition $-2a\leqslant ct\leqslant 2a$. This implies that $C_{1,2}(t)=0$ for t outside the interval $[-2a/c,2a/c]$. To perform the spatial integration in Eq. (9), we make a change of variable from the Cartesian coordinate to a coordinate system adapted to the hyperbola in Fig. 2

$$\begin{cases} x=a\sin(\theta)\cosh(\varphi)\\ y=a\cos(\theta)\sinh(\varphi)\cos(\psi)\\ z=a\cos(\theta)\sinh(\varphi)\sin(\psi)\end{cases} \text{ with } \begin{cases}\varphi\in\mathcal{R}^+\\ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right].\\ \psi\in[0,2\pi]\end{cases} \tag{11}$$

The Jacobian of this change of variable is

$$J(\varphi,\theta,\psi)=a^3\cos(\theta)\sinh(\varphi)[\cosh^2(\varphi)-\sin^2(\theta)]. \tag{12}$$

Then, we have for any point $\mathbf{r}_s$ defined by the coordinates (φ,θ,ψ) $|\mathbf{r}_2-\mathbf{r}_s|=a(\cosh(\varphi)+\sin(\theta))$ and $|\mathbf{r}_1-\mathbf{r}_s|=a(\cosh(\varphi)-\sin(\theta))$, from which it follows

$$\langle C_{1,2}(t)\rangle=2\pi aQ^2T\nu\int_0^{+\infty}\sinh(\varphi)d\varphi\int_{-\pi/2}^{\pi/2}\cos(\theta)\times\delta\left(t+\frac{2a\sin(\theta)}{c}\right)d\theta. \tag{13}$$

A last change of variable $a\sin(\theta)=x$ finally gives

$$\langle C_{1,2}(t)\rangle=2\pi Q^2T\nu\int_0^{+\infty}\sinh(\varphi)d\varphi\int_{-a}^{a}\delta\left(t+\frac{2x}{c}\right)dx. \tag{14}$$

Knowing that the integration of a Dirac function $\delta(t)$ yields a Heaviside step function $H(t)$, the second integral in Eq. (14) gives a rectangle function

$$\Pi(t)=H\left(t+\frac{2a}{c}\right)-H\left(t-\frac{2a}{c}\right), \tag{15}$$

whose amplitude is 1 between $-2a/c$ and $2a/c$ and 0 elsewhere. The first integral (over φ) in Eq. (14) yields the amplitude of the noise correlation function. The limits of the integral in φ have to be defined to prevent $\langle C_{12}(t)\rangle$ from diverging. Actually, φ is the curvilinear coordinate along each hyperbola. In Fig. 2, $\varphi=0$ is the intersection of the hyperbola with the x axis, while the asymptotic branch of the hyperbola corresponds to $\varphi\to+\infty$. Integrating over φ between 0 and φ_0 corresponds then to the measure of the length of each hyperbola on this interval. In Fig. 2, the points verifying $\varphi=\varphi_0$ describe an ellipse that is orthogonal to each of the hyperbola. Interestingly enough, this ellipse (or ellipsoid in 3D) is such that $|\mathbf{r}_2-\mathbf{r}_s|+|\mathbf{r}_1-\mathbf{r}_s|=ct$, i.e., the noise sources whose cumulated travel to A and B are the same. We will see in the next section that this ellipse corresponds to the points that have suffered from the same attenuation in the correlation process. Defining the $\varphi=\varphi_0$ ellipsoid as the 3D compact support on which the integration in Eq. (14) is performed leads to the final result

$$\langle C_{1,2}(t)\rangle=2\pi Q^2T\nu[\cosh(\varphi_0)-1]\Pi(t). \tag{16}$$

The time derivative of the average correlation function is then

$$\frac{d}{dt}\langle C_{1,2}(t)\rangle=4\pi aQ^2T\nu[\cosh(\varphi_0)-1]\times\left[\frac{1}{2a}\delta\left(t+\frac{2a}{c}\right)-\frac{1}{2a}\delta\left(t-\frac{2a}{c}\right)\right]. \tag{17}$$

The Dirac functions in Eq. (17) are the causal and anticausal (or time-reversed) Green's functions from A to B. There is no problem in obtaining a time-reversed expression of the Green's function because a correlation function is defined for negative and positive times. Physically speaking, the time symmetry of Eq. (17) results from our hypothesis of spatially uniform ambient noise distribution. Noise sources surrounding the receivers in A and B, the correlation function contains both propagation information from A to B and B to A. Taking the time derivative of the correlation function, the time symmetry means that both the forward and backward Green's functions between A and B are retrieved from ambient noise cross correlation. The amplitude of these Green's functions is driven by the noise excitation power and the area of the ellipsoidal compact support on which noise sources are taken. Note again that the compact support is necessary in this analysis to prevent the noise correlation function from diverging.

III. FREE SPACE WITH ATTENUATION

One remaining question is the physical meaning of the boundary limit $\varphi=\varphi_0$ necessary to perform the integration in Eq. (14). Actually, φ_0 is strongly related to the presence of attenuation in the medium. Up to now, we have considered a homogeneous propagation medium without attenuation. In that case, the amplitude contribution from noise sources far away from the receivers A and B is not lowered, which leads to a diverging correlation function as seen in Eq. (14). The goal of this section is to show that the presence of attenuation in the medium solves this problem. Volume attenuation is included in the medium by adding an imaginary component to the sound speed $c=c_0+ic_i$, with $c_i \ll c_0$. We chose this usual way to account for attenuation because of its convenience to pursue the mathematical derivation. The frequency dependence of the Green's function that results from attenuation is discussed later. The Green's function is then modified as follows:

$$G(\mathbf{r}_2,t;\mathbf{r}_1,0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\,\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}\exp\left[i\omega\left(t-\frac{|\mathbf{r}_2-\mathbf{r}_1|}{c_0}\right)\right]\times\exp\left(-\omega c_i\frac{|\mathbf{r}_2-\mathbf{r}_1|}{c_0^2}\right). \tag{18}$$

Assuming again that $\langle S(\mathbf{r}_s,t_s)S(\mathbf{r}_{s'},t_{s'})\rangle=Q^2\delta(t_s-t_{s'})\delta(\mathbf{r}_s-\mathbf{r}_{s'})$ and using the same development as in Eqs. (4) to (8), we have now

$$\langle C_{1,2}(t)\rangle=\frac{Q^2}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{\tau}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{d\mathbf{r}_s dt_s d\omega d\omega' d\tau}{|\mathbf{r}_2-\mathbf{r}_s||\mathbf{r}_1-\mathbf{r}_s|}\times\exp\left[i\omega\left(\tau-t_s-\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c_0}\right)+i\omega'\times\left(t+\tau-t_s-\frac{|\mathbf{r}_2-\mathbf{r}_s|}{c_0}\right)\right]\times\exp\left[-c_i\left(\omega\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c_0^2}+\omega'\frac{|\mathbf{r}_2-\mathbf{r}_s|}{c_0^2}\right)\right]. \tag{19}$$

After a change of variable $\tau-t_s=\tau'$, we perform the integration on τ' knowing that $\int_0^{+\infty}\exp[i(\omega+\omega')\tau']d\tau'=\delta(\omega+\omega')$, which leads to

$$\langle C_{1,2}(t)\rangle=\frac{Q^2}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{d\tau d\mathbf{r}_s d\omega}{|\mathbf{r}_2-\mathbf{r}_s||\mathbf{r}_1-\mathbf{r}_s|}\times\exp\left[i\omega\left(t+\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c_0}-\frac{|\mathbf{r}_2-\mathbf{r}_s|}{c_0}\right)\right]\times\exp\left[-\frac{\omega c_i}{c_0^2}(|\mathbf{r}_1-\mathbf{r}_s|+|\mathbf{r}_2-\mathbf{r}_s|)\right]. \tag{20}$$

As in Sec. II, the integral over $d\tau$ is changed into the product $T\nu$. It corresponds to the accumulation of noise sources with a creation rate ν ($\mathrm{m}^{-3}\,\mathrm{s}^{-1}$) over the finite duration signals of length T recorded in A and B. Then, we have

$$\langle C_{1,2}(t)\rangle=\frac{Q^2T\nu}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{d\mathbf{r}_s d\omega}{|\mathbf{r}_2-\mathbf{r}_s||\mathbf{r}_1-\mathbf{r}_s|}\times\exp\left[i\omega\left(t+\frac{|\mathbf{r}_1-\mathbf{r}_s|}{c_0}-\frac{|\mathbf{r}_2-\mathbf{r}_s|}{c_0}\right)\right]\times\exp\left[-\frac{\omega c_i}{c_0^2}(|\mathbf{r}_1-\mathbf{r}_s|+|\mathbf{r}_2-\mathbf{r}_s|)\right]. \tag{21}$$

Equation (21) is the equivalent of Eq. (9) in the presence of attenuation. We recognize in the two exponentials the hyperbola spatial dependence for the phase term and the ellipse spatial dependence for the amplitude term (Fig. 2). We apply then the change of variable done in Sec. II [Eq. (11)] to decouple the phase term from the amplitude term in Eq. (21). It follows

$$\langle C_{1,2}(t)\rangle=\frac{Q^2T\nu}{2\pi}\int_{-\infty}^{+\infty}d\omega\int_0^{+\infty}d\varphi\,\sinh(\varphi)\times\exp\left(-\frac{2a\cosh(\varphi)\omega c_i}{c_0^2}\right)\int_{-a}^{+a}dx\times\exp\left[i\omega\left(t+\frac{2x}{c_0}\right)\right]. \tag{22}$$

The presence of attenuation in the medium makes the integral over φ converge as

$$\int_0^{+\infty}d\varphi\,\sinh(\varphi)\exp\left(-\frac{2a\cosh(\varphi)\omega c_i}{c_0^2}\right)=\frac{c_0^2}{2a\omega c_i}\exp\left(-\frac{2a\omega c_i}{c_0^2}\right), \tag{23}$$

while the integral over x gives

$$\int_{-a}^{+a}dx\,\exp\left[i\omega\left(t+\frac{2x}{c_0}\right)\right]=\frac{c_0}{2i\omega}\left[\exp\left(i\omega\left(t+\frac{2a}{c_0}\right)\right)-\exp\left(i\omega\left(t-\frac{2a}{c_0}\right)\right)\right]. \tag{24}$$

Finally, combining Eqs. (23) and (24), it follows

$$\langle C_{1,2}(t)\rangle=\frac{Q^2T\nu c_0^3}{8\pi a c_i}\int_{-\infty}^{+\infty}d\omega\,\frac{1}{i\omega}\left[\exp\left(i\omega\left(t+\frac{2a}{c_0}\right)\right)-\exp\left(i\omega\left(t-\frac{2a}{c_0}\right)\right)\right]\frac{1}{\omega}\exp\left(-\frac{2a\omega c_i}{c_0^2}\right). \tag{25}$$

From the result derived in Sec. I, we know that the time derivative of the noise correlation function yields the Green's function. In the case of a medium with attenuation, the time derivative of the correlation function gives

$$\frac{d}{dt}\langle C_{1,2}(t)\rangle = \frac{Q^2 T\nu c_0^3}{4\pi c_i}\left\{\int_{-\infty}^{+\infty}\frac{d\omega}{\omega}\frac{1}{2a}\exp\left(i\omega\left(t+\frac{2a}{c_0}\right)\right)\right.$$
$$\times\exp\left(-\frac{2a\omega c_i}{c_0^2}\right)-\int_{-\infty}^{+\infty}\frac{d\omega}{\omega}\frac{1}{2a}$$
$$\left.\times\exp\left(i\omega\left(t-\frac{2a}{c_0}\right)\right)\exp\left(-\frac{2a\omega c_i}{c_0^2}\right)\right\}. \tag{26}$$

The two integrals in Eq. (26) contain a frequency-dependent term $1/\omega$ that prevents us from directly identifying these integrals as the exact Green's function from A to B and B to A. Actually, this $1/\omega$ is due to the choice made to describe the attenuation in the medium. Adding an imaginary part to the sound speed ($c=c_0+ic_i$) means that we made the assumption of a linear dependence of the attenuation with frequency, as shown in the Green's function formulation [Eq. (18)]. If we had chosen a ω^n dependence of the attenuation in the Green's function, we would have obtained a $1/\omega^n$ amplitude term in the derivative of the correlation function [Eq. (26)]. Physically speaking, this means that the attenuation in the medium will impact the estimate of the Green's function from the ambient noise correlation function. More precisely, attenuation acts as a low-pass filter whose frequency behavior follows the frequency dependence of the attenuation in the medium. Thus, the physical interpretation of Eq. (26) is still the same as Eq. (17)

$$\frac{d}{dt}\langle C_{1,2}(t)\rangle \approx 4\pi a Q^2 T\nu\Omega[G(\mathbf{r}_2,0;\mathbf{r}_1,-t)$$
$$-G(\mathbf{r}_1,t;\mathbf{r}_2,0)], \tag{27}$$

where the $\approx$ sign means that the Green's function has been low-pass filtered according to the frequency dependence of the medium attenuation. Comparing Eqs. (26) and (18), we note that the amplitude term Ω is directly related to the formulation of the attenuation term used in of the definition of the Green's function.

We see from the mathematical developments made in Sec. III that the introduction of a small attenuation in the medium makes the correlation function converge without any constraint on the noise source statistics. The final result is the same as in Sec. II, except that the frequency filtering occurs when a frequency-dependent attenuation is introduced. The derivative of the ambient noise correlation function gives birth to a causal and anticausal (or time-reversed) estimate of the Green's function between the two points at which noise has been recorded.

IV. DISCUSSION

In the literature, most experimental results have been obtained by using the noise correlation function (and not its derivative) as a close estimation of the Green's function. Only Weaver's results in ultrasonic reverberant cavities[1] have clearly demonstrated, both theoretically and experimentally, that the Green's function would be retrieved from the diffuse noise correlation function derivative. In similar works

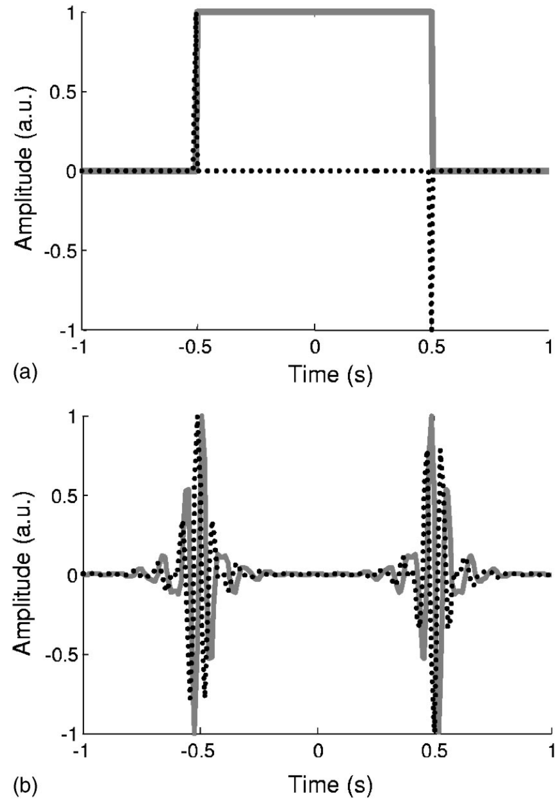

FIG. 3. Representation of the noise correlation function (gray line) and its time derivative (black dotted line): (a) in an infinite bandwidth case, with $a=0.25$ m and $c=1$ m/s; (b) with the same parameters, in a limited bandwidth case [10–20 Hz].

in ultrasonics,[8] underwater acoustics,[2] and geophysics,[4] the correlation function is preferred to its derivative to approximate for the Green's function. Indeed, performing a time derivative on experimental data is usually avoided because it could be the source of strong undesirable noise. However, the mathematical demonstration above clearly shows that it is the derivative of the ambient noise correlation function that converges to the Green's function. What do we lose when we don't perform the time derivative?

An element of response is given in Fig. 3, where ambient noise correlation functions $\langle C_{1,2}(t)\rangle$ are plotted versus their derivatives for a infinite bandwidth [Fig. 3(a)] and a limited bandwidth signal [Fig. 3(b)], respectively. The two functions look very different in the infinite bandwidth case, mostly because the zero-frequency component creates the plateau of the correlation function. However, this dc component will usually not be available in realistic experiments. In the case of a finite bandwidth problem, we see that the two functions resemble each other. Their principal difference is a $\pi/2$ phase shift that does not affect the overall shape of the waveform but that could be of importance in the case of tomography applications where exact arrival times need to be estimated. However, if undesired noise becomes an experimental issue when performing the time derivative of the correlation function, it is not a bad approximation to estimate the Green's function as the noise correlation function itself.

Finally, how could the theoretical demonstration done in Sec. III be adapted to the case of a heterogeneous medium? For example, if we assume a spatial dependence of the sound speed $c_0=c_0(x,y,z)$, is the final result in Eq. (27) still correct? It is possible to show that, for uncorrelated noise sources, Eq. (21) is still valid if the arrival times $|\mathbf{r}_1-\mathbf{r}_s|/c_0$ and $|\mathbf{r}_2-\mathbf{r}_s|/c_0$ are changed into a more general formulation $\int_{\mathbf{r}_s}^{\mathbf{r}_1} ds/c(s)$ and $\int_{\mathbf{r}_s}^{\mathbf{r}_2} ds'/c(s')$, where the paths S and S' between the noise source in $\mathbf{r}_s$ and the receivers in A and B are given by the Fermat principle. In this case, the noise sources that contribute to the noise correlation function at a given time t have to satisfy the following equation:

$$\int_{\mathbf{r}_s}^{\mathbf{r}_1} \frac{ds}{c(s)} - \int_{\mathbf{r}_s}^{\mathbf{r}_2} \frac{ds'}{c(s')} = t. \tag{28}$$

In general, the noise sources that satisfy Eq. (28) are no longer located on a hyperbola as in Fig. 2 and the change of variable done in Eq. (11) is now irrelevant. However, the physical insight derived from the geometrical interpretation in Fig. 2 is still correct. Actually, two conditions are required for this to be true. First, there must exist, for a given t, a 3D surface made of the noise sources that satisfy Eq. (28), and this set of 3D surfaces must cover the whole 3D space when $-2a \leqslant ct \leqslant 2a$. Second, one point in space must belong to one and only one of these 3D surfaces. Those two conditions mean that there exists an isomorphism between the hyperboloids in the homogeneous space case and the 3D sheets in the heterogeneous medium. If this is so, a conformal transformation could be used to shift from the heterogeneous space to the homogeneous one, in which the integration from Eq. (21) to Eq. (27) is performed before the inverse conformal transformation is made to go back to the heterogeneous space. The existence of such a conformal transformation ensures that we have again in a heterogeneous space

$$\frac{d}{dt}\langle C_{1,2}(t)\rangle \propto G(\mathbf{r}_2,0;\mathbf{r}_1,-t)-G(\mathbf{r}_1,t;\mathbf{r}_2,0). \tag{29}$$

However, such a conformal transformation is not always likely to exist. For example, it is well known in underwater acoustics that refraction could generate multiple arrival times between two points. If a noise source creates multiple echoes in A and B, it means that it belongs to different t-invariant 3D surfaces of the correlation function. There is then no isomorphism between the hyperbola in the homogeneous medium and the t-invariant sheets in the refractive medium. In this case, further analysis is necessary to understand the relationship between the noise correlation function and the Green's function.[5,12]

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