

Spatial correlation in arbitrary noise fields with application to ambient sea noise*

Henry Cox

University of California, San Diego, Marine Physical Laboratory of the Scripps Institution of Oceanography, San Diego, California 92132

(Received 30 April 1973)

Explicit expressions are developed for the cross-spectral density between pairs of sensors and the wavenumber-frequency spectrum projected onto the line joining these sensors, when they are placed in arbitrary positions in noise fields which are described by an arbitrary directional distribution of uncorrelated plane waves. The approach is to expand the directional distribution in spatial harmonics. Each harmonic leads to a corresponding term, with the same coefficient, in series representations for the cross-spectral density function and the wavenumber-frequency spectrum. The approach is particularly attractive when the field can be adequately represented by a relatively small number of harmonics. Both two- and three-dimensional fields are considered. The approach is applied to the vertical directionality of ambient sea noise and is related to some existing models of ambient sea noise. Some new models are presented. Results are compared with reported experimental data and found to be in good agreement over a wide frequency range.

Subject Classification: 13.9.

INTRODUCTION

The performance of an array of sensors in a noise field depends on the cross-correlation functions or equivalently the cross-spectral density functions which describe the second-order statistical relationships between all pairs of sensors in the array. These functions in turn depend on the directional properties of the noise field and the geometry of the array. A common model represents a noise field as a sum of uncorrelated plane waves propagating from various directions. Such a field is usually described by a directional density function which indicates the amount of noise power coming from each direction. While such a description is intuitively appealing, it does not provide convenient expressions for the second-order statistical properties which are of interest in many applications.

The purposes of this paper are twofold: first, to develop explicit expressions for the second-order statistical relationships between the outputs of pairs of omnidirectional sensors located in arbitrary positions in a noise field which is described by an arbitrary directional density of uncorrelated plane waves; second, to apply the results to the problem of vertical directionality of ambient sea noise. This application is intended both to illustrate the method and to present new results which have some importance of their own.

In isotropic noise fields the directional density of the noise is uniform so that the same amount of noise arrives from all directions. In a three-dimensional isotropic noise field it is well known¹ that the normalized cross-spectral density between two sensors separated by a distance s is $[\sin(\omega s/c)](\omega s/c)^{-1}$, where c is the velocity of propagation. In a two-dimensional isotropic noise field² the normalized cross-spectral density is

$J_0(\omega s/c)$ where J_0 is the zero-order Bessel function of the first kind.

While these results for isotropic noise have been known for many years, there has been relatively little work in describing the properties of directional noise fields. The problem of vertical directionality of ambient noise in the ocean is an exception in this regard and considerable experimental³⁻⁸ and theoretical⁹⁻¹⁵ work has been published in the last decade. The paper by Cron and Sherman⁹ is apparently the first work to present an analytical expression for the cross-correlation function between pairs of sensors in a directional noise field. They model the surface of the ocean as a uniform distribution of independent directional noise sources. For the situation in which the power directivity of these individual sources is of the form $\cos^{2m}(\alpha)$, where α is measured from the normal to the surface, they present explicit expressions for the correlation between deep sensors displaced either vertically or horizontally from each other. Their model is equivalent to a superposition of independent plane waves with a directional density which is nonzero over the upper hemisphere and zero over the lower hemisphere so that all the waves travel downward. Liggett and Jacobson¹¹ suggest a single parameter family of directional density functions which similarly are nonzero only in the upper hemisphere and also present expressions for the correlation functions of vertically displaced deep sensors.

In this paper, both the directional distribution of uncorrelated plane waves and the orientation of the sensors are arbitrary. The approach is to represent the directional density function in terms of spatial harmonics. Each spatial harmonic of the directional density function leads to corresponding terms in the

cross-spectral density between pairs of sensors and in the projected wavenumber-frequency spectrum, where the projection of wavenumber is onto the line joining the two sensors. Moreover the coefficients of these corresponding terms are the same. This representation is particularly attractive when the directional density function may be described by a relatively small number of harmonics so that the desired second-order statistics are given as the sum of a correspondingly small number of terms.

Since the equations for two-dimensional fields are less involved, we will first develop the theoretical results for two-dimensional fields in considerable detail showing alternative approaches so that basic ideas may be easily understood. The theoretical results for three-dimensional fields will then be presented without as much detail in derivation since the approach is analogous to that for two-dimensional fields.

The theory of three-dimensional fields will then be related to the ambient noise models of Cron and Sherman⁹ and Liggett and Jacobson.¹¹ Some new models will then be presented which allow the second-order statistics of interest to be computed with ease. Finally, these new models will be compared with reported experimental data covering a wide range of frequencies.

I. DEFINITIONS

Consider two omnidirectional sensors in a temporally stationary noise field. The temporal cross-correlation function between the sensor outputs may be defined as follows:

$$q_{12}(\tau) = \overline{n_1(t)n_2(t-\tau)}, \tag{1}$$

where the overbar denotes ensemble average. The cross-spectral density function is defined through the following Fourier transformation¹⁶

$$Q_{12}'(\omega) = \int q_{12}(\tau) \exp(-i\omega\tau) d\tau, \tag{2}$$

with the inverse relationship

$$q_{12}(\tau) = (2\pi)^{-1} \int Q_{12}'(\omega) \exp(i\omega\tau) d\omega. \tag{3}$$

The prime in $Q_{12}'(\omega)$ is to distinguish it from the normalized cross-spectral density which will be introduced shortly.

The noise fields of interest in this paper are composed of a superposition of uncorrelated plane waves arriving at the sensors from various directions. The approach which will be used in the analysis of such fields involves integration of the effect of a single plane wave weighted by the directional distribution of plane waves. Thus it is useful initially to consider the effect of a single plane wave.

Let \mathbf{p}_1 and \mathbf{p}_2 be position vectors (x, y, z) describing the location of the first and second sensors, respectively, and let \mathbf{u} be a unit vector in the direction opposite to that in which the plane wave is propagating. Then if a plane wave propagates past the sensors the output of the second sensor is a delayed version of the output of the first sensor. That is,

$$n_2(t) = n_1[t - \mathbf{u} \cdot (\mathbf{p}_1 - \mathbf{p}_2)/c], \tag{4}$$

where c is the velocity of propagation. Then, from Eq. 1

$$q_{12}(\tau) = q_{11}[\tau + \mathbf{u} \cdot (\mathbf{p}_1 - \mathbf{p}_2)/c], \tag{5}$$

and

$$Q_{12}'(\omega) = Q_{11}'(\omega) \exp\{i\mathbf{k} \cdot (\mathbf{p}_1 - \mathbf{p}_2)\}, \tag{6}$$

where

$$\mathbf{k} = (\omega/c)\mathbf{u}. \tag{7}$$

Notice that $Q_{12}'(\omega)$ depends on the separation of the sensors and the orientation of the sensors relative to the direction from which the wave is propagating but not on the absolute positions of the sensors. A field with these properties is called spatially homogeneous. Fields which are composed of a superposition of uncorrelated plane waves are spatially homogeneous. In such a field $Q_{12}'(\omega)$ is invariant under translations but not in general under rotations of the positions of the sensors. A special case of a homogeneous field is the isotropic noise field which results from a uniform distribution of plane waves over all directions. In an isotropic field $Q_{12}'(\omega)$ is also invariant under rotations of the sensors.

In a homogeneous noise field the power spectral density is the same at every point in the field. Hence, it is convenient to define a normalized cross-spectral density as follows:

$$Q_{12}(\omega) = Q_{12}'(\omega)/Q_{11}'(\omega) = Q_{12}'(\omega)/Q_{22}'(\omega). \tag{8}$$

The general definitions of Eqs. 1 and 2 will now be made more specific by explicitly including the positions of the sensors in the definitions.

In a homogeneous field without loss of generality the coordinate system may be chosen so that the line joining the two sensors passes through the origin. It then becomes convenient to work in spherical or cylindrical coordinates. We shall use the following definitions for coordinate systems:

Spherical coordinates	Cylindrical coordinates
$x = r \sin\theta \cos\phi,$ $y = r \sin\theta \sin\phi,$ $z = r \cos\theta,$	$x = r \cos\phi,$ $y = r \sin\phi$

so that ϕ is the azimuthal angle and θ is the angle, in spherical coordinates, measured from the zenith. The

space-time correlation function between two sensors in a three-dimensional field may then be written as follows:

$$q(s, \tau; \gamma, \xi) = \overline{n(r, \gamma, \xi, t)n(r-s, \gamma, \xi, t-\tau)}, \quad (9)$$

where $n(r, \gamma, \xi, t)$ is the output of a sensor located at $(r, \theta = \gamma, \phi = \xi)$ at time t . The angles γ and ξ in Eq. 9 describe the orientation of the line joining the sensors. The separation of the sensors is s , and τ is a time delay.

Taking the temporal Fourier transform of Eq. 9 yields

$$Q'(s, \omega; \gamma, \xi) = \int q(s, \tau; \gamma, \xi) \exp(-i\omega\tau) d\tau; \quad (10)$$

$Q'(s, \omega; \gamma, \xi)$ is the cross-spectral density between the outputs of two sensors with a separation s on the line (γ, ξ) . It is a function which arises frequently in the analysis of noise fields.

Taking the spatial Fourier transform of $Q'(s, \omega; \gamma, \xi)$ results in the wavenumber-frequency spectrum

$$\tilde{Q}'(k, \omega; \gamma, \xi) = \int Q'(s, \omega; \gamma, \xi) \exp(-iks) ds, \quad (11)$$

or from Eq. 10,

$$\tilde{Q}'(k, \omega; \gamma, \xi) = \int \int q(s, \tau; \gamma, \xi) \times \exp[-i(ks + \omega\tau)] ds d\tau. \quad (12)$$

The inverse transform is

$$q(s, \tau; \gamma, \xi) = (2\pi)^{-2} \iint \tilde{Q}'(k, \omega; \gamma, \xi) \times \exp[i(ks + \omega\tau)] dk d\omega. \quad (13)$$

The quantity $\exp[i(ks + \omega\tau)]$ in Eq. 13 represents a wave whose apparent wavelength on the line (γ, ξ) is $2\pi/k$. Thus k in Eq. 13 is the projection of the wavenumber vector \mathbf{k} onto the line (γ, ξ) . Hence $|k| \leq |\mathbf{k}| = \omega/c$. The projection operation relates the direction of propagation to what is observed along the line (γ, ξ) . Thus, $q(s, \tau; \gamma, \xi)$ in Eq. 13 may be interpreted as a sum of such plane waves weighted by the wavenumber-frequency spectrum $\tilde{Q}'(k, \omega; \gamma, \xi)$ which describes the density of plane waves in the interval $dk d\omega$.

For two-dimensional fields, all definitions remain the same if the parameters γ and θ are dropped.

It is convenient to define the normalized quantities

$$Q(s, \omega; \gamma, \xi) = Q'(s, \omega; \gamma, \xi) / Q'(\omega) \quad (14)$$

and

$$\tilde{Q}(k, \omega; \gamma, \xi) = \tilde{Q}'(k, \omega; \gamma, \xi) / Q'(\omega), \quad (15)$$

where $Q'(\omega)$ is the spectral density at all points in the

field. This normalization separates the dependence of the spatial characteristics on frequency from the dependence of the overall noise level on frequency. In this paper we will be primarily interested in the spatial aspects of the problem and hence $Q(s, \omega; \gamma, \xi)$ and $\tilde{Q}(k, \omega; \gamma, \xi)$ will be studied in some detail. These normalized quantities are spatial Fourier transform pairs since division by $Q'(\omega)$ does not affect Eq. 11. The space-time correlation function may be obtained from either of these by Fourier transformation once $Q'(\omega)$ is specified.

For example, for a monochromatic field,

$$Q'(\omega) = \pi P [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)],$$

where δ is the Dirac function and P is the noise power. In such a field the normalized space-time correlation function defined as $\rho(s, \tau; \gamma, \xi) = q(s, \tau; \gamma, \xi) / q(0, 0; \gamma, \xi)$ is given by

$$\rho(s, \tau; \gamma, \xi) = \text{Re}[Q(s, \omega_0; \gamma, \xi)] \cos \omega_0 \tau - \text{Im}[Q(s, \omega_0; \gamma, \xi)] \sin \omega_0 \tau. \quad (16)$$

In the special case of zero time delay, this reduces to

$$\rho(s, 0; \gamma, \xi) = \text{Re}[Q(s, \omega_0; \gamma, \xi)]. \quad (17)$$

II. TWO-DIMENSIONAL FIELDS

A two-dimensional noise field consisting of a superposition of uncorrelated plane waves from various azimuthal directions may be described by a normalized azimuthal density function $G(\phi, \omega)$. The function $G(\phi, \omega)$ is nonnegative and real, representing the relative density of plane waves traveling from the direction ϕ at frequency ω . The normalization is such that

$$(2\pi)^{-1} \int_{-\pi}^{\pi} G(\phi, \omega) d\phi = 1. \quad (18)$$

The problem is to determine the cross-spectral density between any two points in such a field.

Since the field is homogeneous, the origin of the cylindrical coordinate system may be chosen so that the first sensor is located at (r, ξ) and the second sensor is located at $(r-s, \xi)$. Thus the sensors are on the line $\phi = \xi$, separated by a distance s . This situation is illustrated in Fig. 1.

For a single plane wave from a direction ϕ_0 , $G(\phi, \omega) = 2\pi\delta(\phi - \phi_0)$. Thus, from Eq. 6 the normalized cross-spectral density at the two sensors due to a single plane wave is $\exp[i(\omega/c)s \cos(\phi_0 - \xi)]$ and $Q(s, \omega; \xi)$ may be obtained by integrating the effect of a single plane wave over all azimuthal directions weighted by the

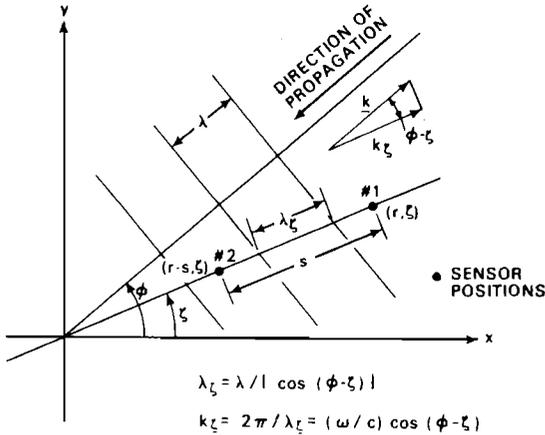


FIG. 1. Illustration of sensor locations and a single plane wave in two dimensions.

azimuthal density of plane waves; that is,

$$Q(s, \omega; \zeta) = (2\pi)^{-1} \int_{-\pi}^{\pi} G(\phi, \omega) \times \exp[i(\omega/c)s \cos(\phi - \zeta)] d\phi. \quad (19)$$

Any density function $G(\phi, \omega)$ may be represented by a cylindrical harmonic expansion of the form

$$G(\phi, \omega) = \sum_{m=0}^{\infty} \epsilon_m [a_m(\omega) \cos(m\phi) + b_m(\omega) \sin(m\phi)], \quad (20)$$

where $\epsilon_m = 1$ for $m = 0$ and $\epsilon_m = 2$ for $m = 1, 2, 3, \dots$, and $a_m(\omega)$ and $b_m(\omega)$ are Fourier coefficients given by the following:

$$a_m(\omega) = (2\pi)^{-1} \int_{-\pi}^{\pi} G(\phi, \omega) \cos(m\phi) d\phi \quad (21)$$

and

$$b_m(\omega) = (2\pi)^{-1} \int_{-\pi}^{\pi} G(\phi, \omega) \sin(m\phi) d\phi. \quad (22)$$

Because of the normalization of Eq. 18, $a_0(\omega)$ is always unity.

Since $\cos(\phi - \zeta)$ appears in the exponential in Eq. 19, it is convenient to rewrite Eq. 20 as follows:

$$G(\phi, \omega) = \sum_{m=0}^{\infty} \epsilon_m \{ c_m(\zeta; \omega) \cos[m(\phi - \zeta)] + d_m(\zeta; \omega) \sin[m(\phi - \zeta)] \}, \quad (23)$$

where the coefficients in Eq. 23 are related to the coefficients in Eq. 20 by the following rotation:

$$c_m(\zeta; \omega) = a_m(\omega) \cos(m\zeta) + b_m(\omega) \sin(m\zeta) \quad (24)$$

and

$$d_m(\zeta; \omega) = b_m(\omega) \cos(m\zeta) - a_m(\omega) \sin(m\zeta). \quad (25)$$

Notice that $c_0(\zeta, \omega) = 1$.

Substituting from Eq. 23 into Eq. 19 and letting $\alpha = \phi - \zeta$ yields the following:

$$Q(s, \omega; \zeta) = \sum_{m=0}^{\infty} (2\pi)^{-1} \int_{-\pi}^{\pi} \epsilon_m [c_m(\zeta; \omega) \cos(m\alpha) + d_m(\zeta; \omega) \sin(m\alpha)] \exp[i(\omega/c)s \cos \alpha] d\alpha. \quad (26)$$

The $\sin(m\alpha)$ terms in Eq. 26 integrate to zero. Using the relationship¹⁷

$$i^m J_m(z) = (2\pi)^{-1} \int_{-\pi}^{\pi} \cos(m\alpha) \exp(iz \cos \alpha) d\alpha, \quad (27)$$

where $J_m(z)$ is the m th-order Bessel function of the first kind, Eq. 26 may be integrated to obtain

$$Q(s, \omega; \zeta) = \sum_{m=0}^{\infty} i^m \epsilon_m c_m(\zeta; \omega) J_m(\omega s/c). \quad (28)$$

Equation 28 expresses $Q(s, \omega; \zeta)$ as a sum of Bessel functions $J_m(\omega s/c)$ which depend on the separation of the sensors s weighted by the Fourier coefficients of the cylindrical harmonic expansion of the azimuthal density function. The orientation of the sensors enters Eq. 28 through the dependence of $c(\zeta; \omega)$ on the angle ζ which the line joining the sensors makes with the x axis. The factor i^m in Eq. 28 indicates that odd harmonics introduce quadrature components in the cross-spectral density corresponding to a net propagation of noise power along the line joining the sensors.

The simplest special case of Eq. 28 is the isotropic noise field which has only the zero-order harmonic. In this case $G(\phi, \omega) = 2\pi$, so that $c_0 = 1$ is the only nonzero component. Then Eq. 28 yields $Q(s, \omega; \zeta) = J_0(\omega s/c)$, which is the well-known result for the two-dimensional isotropic field.

It is interesting to check that Eq. 28 gives the proper result for a plane wave. Consider a single plane wave from angle ϕ_0 so that $G(\phi, \omega) = 2\pi \delta(\phi - \phi_0)$ and $c_m(\zeta; \omega) = \cos[m(\phi_0 - \zeta)]$. Then Eq. 28 yields the following known series expansion for $\exp[i(\omega s/c) \cos(\phi_0 - \zeta)]$:

$$\sum_{m=0}^{\infty} \epsilon_m i^m \cos[m(\phi_0 - \zeta)] J_m(\omega s/c) = \exp[i(\omega s/c) \cos(\phi_0 - \zeta)]. \quad (29)$$

Since the series of Eq. 29 has an infinite number of nonzero terms, the spatial harmonic expansion is not a very efficient way of representing a single plane wave. This series expansion does suggest an alternative derivation of Eq. 28, namely substituting the series expansion of Eq. 29 directly into Eq. 19 and interchanging the order of summation and integration.

Although Eq. 28 is a very general result, its utility is most evident when realistic noise fields may be approximated by a relatively small number of cylindrical harmonics so that Eq. 28 has a correspondingly

small number of nonzero terms. A fairly broad class of noise fields may be represented by a convex combination of an isotropic component and a directional component of the following form:

$$G(\phi, \omega) = \beta(\omega) + [1 - \beta(\omega)]D(\phi, \omega), \quad 0 \leq \beta(\omega) \leq 1, \\ D(\phi, \omega) \geq 0, \quad (30)$$

where $D(\phi, \omega)$ is a suitable normalized directional density function. For example, $D(\phi, \omega)$ could be chosen to be proportional to functions with finite cylindrical harmonic expansions such as

$$\cos^{2n}(\xi) = (2)^{-2n} \sum_{j=0}^n \epsilon_j \binom{2n}{n+j} \cos(2j\xi), \quad (31)$$

where

$$\binom{2n}{n+j}$$

is the binomial coefficient, or

$$\left(\frac{\sin(n\xi)}{n \sin(\xi)} \right)^2 = \frac{1}{n} [1 + (2/n) \sum_{j=1}^{n-1} (n-j) \cos(2j\xi)]. \quad (32)$$

Setting ξ in Eqs. 31 or 32 equal to ϕ results in functions with absolute maximums at both $\phi=0$ and $\phi=\pi$, while setting $\xi = \frac{1}{2}\phi$ yields functions with a single absolute maximum at $\phi=0$. The constant of proportionality is chosen so that Eq. 18 is satisfied. The function $\cos^{2n}(\xi)$ is attractive because it has no sidelobes. The function given in Eq. 32 is attractive because it has a sharper peak for the same number of terms. The use of an isotropic noise component which is about 10 dB below this directional component can cover up the sidelobes. The approach is easily generalized to make $G(\phi, \omega)$ a convex combination of several components.

Recalling that the wavenumber-frequency spectrum $\tilde{Q}(k, \omega; \zeta)$ is the spatial Fourier transform of $Q(s, \omega; \zeta)$ the wavenumber-frequency spectrum along any line $\phi = \zeta$ may be obtained by taking the Fourier transform of Eq. 28. Using the relationship¹⁸

$$\int i^m J_m(z) \exp(-ikz) dz = 2T_m(k) [1 - k^2]^{-\frac{1}{2}}, \quad |k| \leq 1, \\ = 0, \quad |k| > 1, \quad (33)$$

where $T_m(k)$ is the m th-order Chebyshev polynomial of the first kind, we obtain

$$\tilde{Q}(k, \omega; \zeta) = 2(c/\omega) [1 - (ck/\omega)^2]^{-\frac{1}{2}} \\ \times \sum_{m=0}^{\infty} \epsilon_m c_m(\zeta, \omega) T_m(ck/\omega), \quad |k| \leq \omega/c, \\ = 0, \quad |k| > \omega/c. \quad (34)$$

Thus the wavenumber-frequency spectrum is a sum of Chebyshev polynomials weighted by the corresponding coefficient of the harmonic expansion of the azimuthal density function and the factor $[1 - (ck/\omega)^2]^{-\frac{1}{2}}$,

which is the weighting function under which Chebyshev polynomials are orthogonal.

The simplest special case of Eq. 34 is the isotropic field when c_0 is the only nonzero coefficient. Then

$$\tilde{Q}(k, \omega; \zeta) = 2[(\omega/c)^2 - k^2]^{-\frac{1}{2}}, \quad |k| \leq \omega/c, \\ = 0, \quad |k| \geq \omega/c, \quad (35)$$

which is independent of the angle ζ .

An alternative approach to the problem is to determine the normalized wavenumber-frequency spectrum $\tilde{Q}(k, \omega; \zeta)$ directly from the azimuthal density function $G(\phi, \omega)$ and then to obtain $Q(s, \omega; \zeta)$ by Fourier transformation. Proceeding in this manner, we observe that

$$(2\pi)^{-1} \int_{-\omega/c}^{\omega/c} \tilde{Q}(k, \omega; \zeta) dk = (2\pi)^{-1} \int_{-\pi}^{\pi} G(\phi, \omega) d\phi = 1. \quad (36)$$

Let $\alpha = \phi - \zeta$, so that the projected wavenumber on the line $\phi = \zeta$ of a plane wave at frequency ω from direction ϕ is

$$k = (\omega/c) \cos \alpha. \quad (37)$$

Then $\tilde{Q}(k, \omega; \zeta)$ may be obtained from $G(\phi, \omega)$ by a simple change of variables as follows:

$$\tilde{Q}(k, \omega; \zeta) = [G(\alpha, \omega) + G(-\alpha, \omega)] / |dk/d\alpha|, \quad 0 \leq \alpha \leq \pi, \quad (38)$$

or from Eq. 37,

$$\tilde{Q}(k, \omega; \zeta) = \{G[\cos^{-1}(ck/\omega), \omega] + G[-\cos^{-1}(ck/\omega), \omega]\} \\ \times (c/\omega) [1 - (ck/\omega)^2]^{-\frac{1}{2}}, \quad |k| \leq \omega/c. \quad (39)$$

Substituting the harmonic expansion of Eq. 23 into Eq. 39 and recognizing that

$$T_m(ck/\omega) = \cos[m \cos^{-1}(ck/\omega)] \quad (40)$$

results in Eq. 34, which was obtained previously by the alternative approach.

III. THREE-DIMENSIONAL FIELDS

A. General

In a three-dimensional field we define a normalized directional density function $F(\theta, \phi, \omega)$ which describes the density of uncorrelated plane waves from the direction (θ, ϕ) . The normalization is such that

$$(4\pi)^{-1} \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi, \omega) \sin \theta d\theta d\phi = 1. \quad (41)$$

As before, without loss of generality, the origin of the spherical coordinate system is chosen to be on the line joining the sensors of interest. Let the position of the first sensor be $(r, \theta = \gamma, \phi = \zeta)$ and the position of the second sensor be $(r - s, \gamma, \zeta)$. Then the quantity $\mathbf{k} \cdot (\mathbf{p}_1 - \mathbf{p}_2)$ in Eq. 6 becomes

$$\mathbf{k} \cdot (\mathbf{p}_1 - \mathbf{p}_2) = (\omega s/c) [\sin \theta \sin \gamma \cos(\phi - \zeta) \\ + \cos \theta \cos \gamma]. \quad (42)$$

Then $Q(s,\omega; \gamma, \zeta)$ is obtained by integrating over all directions the effect of a single plane wave weighted by the directional density function as follows:

$$Q(s,\omega; \gamma, \zeta) = (4\pi)^{-1} \int_0^{2\pi} \int_0^\pi F(\theta, \phi, \omega) \sin\theta \exp\{i(\omega s/c)\} \times [\sin\theta \sin\gamma \cos(\phi - \zeta) + \cos\theta \cos\gamma] d\theta d\phi, \quad (43)$$

which is analogous to Eq. 19 in the two-dimensional case.

Any directional density function $F(\theta, \phi, \omega)$ may be expanded in terms of spherical surface harmonics of the form

$$F(\theta, \phi, \omega) = \sum_{n=0}^\infty \sum_{m=0}^n [a_n^m(\omega) \cos(m\phi) + b_n^m(\omega) \sin(m\phi)] P_n^m(\cos\theta), \quad (44)$$

where $P_n^m(\cos\theta)$ is the associated Legendre function of the first kind of order m and degree n . The normalization of Eq. 41 makes $a_0^0(\omega) = 1$. Defining $c_n^m(\zeta)$ and $d_n^m(\zeta)$ in analogy with Eqs. 24 and 25 as

$$c_n^m(\zeta, \omega) = a_n^m(\omega) \cos(m\zeta) + b_n^m(\omega) \sin(m\zeta) \quad (45)$$

and

$$d_n^m(\zeta, \omega) = b_n^m(\omega) \cos(m\zeta) - a_n^m(\omega) \sin(m\zeta), \quad (46)$$

equation 44 may be rewritten in the following form:

$$F(\theta, \phi, \omega) = \sum_{n=0}^\infty \sum_{m=0}^n \{c_n^m(\zeta, \omega) \cos[m(\phi - \zeta)] + d_n^m(\zeta, \omega) \sin[m(\phi - \zeta)]\} P_n^m(\cos\theta). \quad (47)$$

Letting $\alpha = \phi - \zeta$, substituting from Eq. 47 into 43, and interchanging the order of summation and integration yields

$$Q(s,\omega; \gamma, \zeta) = \sum_{n=0}^\infty \sum_{m=0}^n \frac{1}{2} \int_0^{2\pi} \left\{ (2\pi)^{-1} \int_0^{2\pi} [c_n^m(\zeta, \omega) \cos m\alpha + d_n^m(\zeta, \omega) \sin m\alpha] \exp[i(\omega s/c) \sin\theta \sin\gamma \cos\alpha] d\alpha \right\} \times \exp[i(\omega s/c) \cos\theta \cos\gamma] P_n^m(\cos\theta) \sin\theta d\theta. \quad (48)$$

The inner integral in Eq. 48 may be integrated using Eq. 27; then Eq. 48 becomes

$$Q(s,\omega; \gamma, \zeta) = \sum_{n=0}^\infty \sum_{m=0}^n (i^m/2) \int_0^\pi c_n^m(\zeta, \omega) J_m[(\omega s/c) \sin\theta \sin\gamma] \times \exp[i(\omega s/c) \cos\theta \cos\gamma] P_n^m(\cos\theta) \sin\theta d\theta. \quad (49)$$

The integral in Eq. 49 may be integrated exactly¹⁹ to

yield the following simple expression for $Q(s,\omega; \gamma, \zeta)$:

$$Q(s,\omega; \gamma, \zeta) = \sum_{n=0}^\infty \sum_{m=0}^n i^n c_n^m(\zeta, \omega) P_n^m(\cos\gamma) j_n(\omega s/c), \quad (50)$$

where $j_n(\omega s/c)$ is the n th-order spherical Bessel function of the first kind. An attractive feature of Eq. 50 is that for each term in the series the dependence on γ, ζ , and s is factored. Thus, for a line array, γ and ζ would be the same for all pairs of sensors and only the spacing in the quantity $j_n(\omega s/c)$ would vary.

The simplest special case of Eq. 50 is for isotropic noise in which case $c_0^0(\zeta, \omega) = 1$ is the only nonzero coefficient. Then

$$Q(s,\omega; \gamma, \zeta) = j_0(\omega s/c) = \frac{\sin(\omega s/c)}{(\omega s/c)}, \quad (51)$$

which is the well-known result for spherically isotropic fields.

The wavenumber-frequency spectrum $\bar{Q}(k,\omega; \gamma, \zeta)$ may be obtained by taking the spatial Fourier transform as in Eq. 11. Using the relationship²⁰

$$\int i^n j_n(z) \exp(-ikz) dz = \pi P_n(k), \quad |k| \leq 1, \\ = 0, \quad |k| > 1, \quad (52)$$

where $P_n(k)$ is the Legendre polynomial of the first kind, the following is obtained from Eq. 50 by Fourier transformation:

$$\bar{Q}(k,\omega; \gamma, \zeta) = \pi(c/\omega) \sum_{n=0}^\infty \sum_{m=0}^n c_n^m(\omega, \zeta) P_n^m(\cos\gamma) P_n^m(ck/\omega), \\ \text{for } |k| \leq \omega/c, \\ = 0, \quad \text{for } |k| > \omega/c. \quad (53)$$

The general results for an arbitrary three-dimensional field composed of a superposition of independent plane waves are given in Eqs. 50 and 53.

In the simplest special case of an isotropic field where $c_0^0 = 1$ is the only nonzero coefficient, Eq. 53 reduces to

$$\bar{Q}(k,\omega; \gamma, \zeta) = \pi c/\omega, \quad \text{for } |k| \leq \omega/c, \\ = 0, \quad \text{for } |k| > \omega/c, \quad (54)$$

so that the wavenumber spectrum is flat over the region $|k| \leq \omega/c$ and is of course independent of the orientation (γ, ζ) of the line.

B. Azimuthally Uniform Fields

1. General Results

An important special case occurs when $F(\theta, \phi, \omega)$ is independent of the azimuthal angle ϕ . This situation

arises in the study of the vertical directionality of ambient noise in the deep ocean. Integrating $F(\theta, \phi, \omega)$ over ϕ yields

$$\frac{1}{2\pi} \int_0^{2\pi} F(\theta, \phi, \omega) d\phi = F(\theta, \omega), \quad (55)$$

where $F(\theta, \omega)$ has the same functional form as $F(\theta, \phi, \omega)$ in the case of azimuthally uniform fields. It follows from Eq. 41 that

$$\frac{1}{2} \int_0^\pi F(\theta, \omega) \sin\theta d\theta = 1. \quad (56)$$

In such a field Eq. 43 may be rewritten as follows:

$$Q(s, \omega; \gamma, \zeta) = \frac{1}{2} \int_0^\pi \left[(2\pi)^{-1} \times \int_0^{2\pi} \exp[i(\omega s/c) \sin\theta \sin\gamma \cos(\phi - \zeta)] d\phi \right] \times F(\theta, \omega) \exp[i(\omega s/c) \cos\theta \cos\gamma] \sin\theta d\theta. \quad (57)$$

The inner integral in Eq. 57 may be integrated using Eq. 27. Then Eq. 57 becomes

$$Q(s, \omega; \gamma, \zeta) = \frac{1}{2} \int_0^\pi J_0[(\omega s/c) \sin\gamma \sin\theta] F(\theta, \omega) \times \exp[i(\omega s/c) \cos\gamma \cos\theta] \sin\theta d\theta, \quad (58)$$

which is independent of the azimuthal angle ζ . The appropriate harmonic expansion for $F(\theta, \omega)$ is the zonal harmonics

$$F(\theta, \omega) = \sum_{n=0}^{\infty} c_n(\omega) P_n(\cos\theta), \quad 0 \leq \theta \leq \pi, \quad (59)$$

which is a simplified form of Eq. 47. $P_n(\cos\theta)$ is a Legendre polynomial of the first kind as before. Substituting from Eq. 59 into Eq. 58 and integrating yields the following simplified form of Eq. 50:

$$Q(s, \omega; \gamma, \zeta) = \sum_{n=0}^{\infty} i^n c_n(\omega) P_n(\cos\gamma) j_n(\omega s/c) \quad (60)$$

for the case of azimuthally uniform fields. Similarly Eq. 53 simplifies to

$$\begin{aligned} \tilde{Q}(k, \omega; \gamma, \zeta) &= \pi(c/\omega) \sum_{n=0}^{\infty} c_n(\omega) P_n(\cos\gamma) P_n(ck/\omega), \quad |k| \leq \omega/c, \\ &= 0, \quad |k| > \omega/c. \end{aligned} \quad (61)$$

Equations 58–61 are the general results for an azimuthally uniform field. Further simplifications occur when the sensors are joined by either a vertical line ($\gamma=0$ or π) or a horizontal line ($\gamma=\frac{1}{2}\pi$). In these situa-

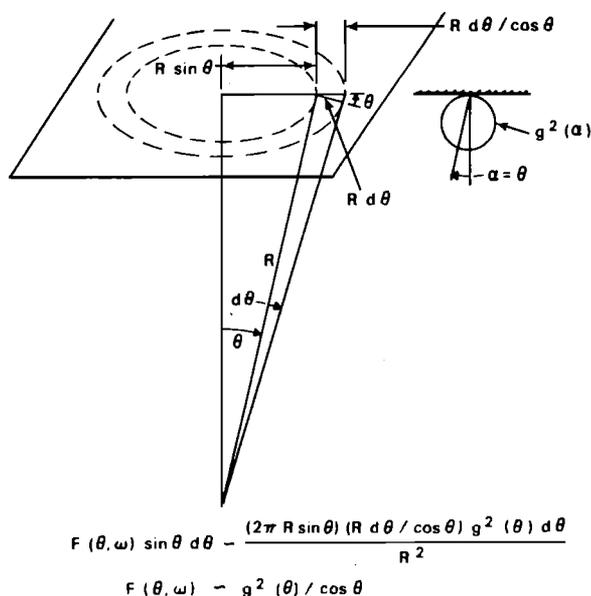


FIG. 2. Relationship of Cron and Sherman surface-noise model to directional density of uncorrelated plane waves.

tions the Legendre polynomials in Eqs. 60 and 61 may be evaluated using the following formulas:

$$P_n(\pm 1) = (\pm 1)^n, \quad (62)$$

$$P_{2n}(0) = (-1)^n (2n)! [2^{2n} (n!)^2]^{-1}, \quad (63a)$$

and

$$P_{2n+1}(0) = 0. \quad (63b)$$

Simplifications also occur in Eq. 58, which becomes

$$Q(s, \omega; 0, \zeta) = \frac{1}{2} \int_0^\pi F(\theta, \omega) \exp[i(\omega s/c) \cos\theta] \sin\theta d\theta \quad (64)$$

for $\gamma=0$, and

$$Q(s, \omega; \pi/2, \zeta) = \frac{1}{2} \int_0^\pi J_0[(\omega s/c) \sin\theta] F(\theta, \omega) \sin\theta d\theta \quad (65)$$

for $\gamma=\frac{1}{2}\pi$.

For the case of a vertical line the wavenumber-frequency spectrum is simply related to $F(\theta, \omega)$ as follows:

$$\tilde{Q}(k, \omega; 0, \zeta) = \pi(c/\omega) F[\cos^{-1}(ck/\omega), \omega], \quad (66)$$

which may be obtained by a simple change of variables using $k = (\omega/c) \cos\theta$.

2. Application to Vertical Distribution of Ambient Sea Noise

a. Relationship to Other Models

Cron and Sherman,⁹ suggest a surface-noise model which consists of a uniform distribution of independent noise sources over the surface of the ocean. Each

source has a power directivity function $g^2(\alpha, \omega)$ where α is the angle measured from the downward-pointing $-z$ axis. When the depth of the sensors is large compared to the spacing between the sensors, it can be seen from Fig. 2 that this model is equivalent to a distribution of independent plane waves with

$$F(\theta, \omega) = K g^2(\theta, \omega) / \cos \theta, \quad 0 \leq \theta \leq \frac{1}{2}\pi, \\ = 0, \quad \frac{1}{2}\pi < \theta \leq \pi, \quad (67)$$

where K is a normalizing constant given by

$$K = 2 \left[\int_0^{\pi/2} g^2(\theta, \omega) \tan \theta d\theta \right]^{-1} \quad (68)$$

so that Eq. 56 is satisfied. Thus, the uniform distribution of independent directional sources on the surface leads to an equivalent directional distribution of plane waves with a directional density function which is zero over the lower hemisphere so that all the waves travel downward. The directional density function over the upper hemisphere is proportional to the power directivity function of the individual sources on the surface divided by $\cos \theta$.

The general result of Cron and Sherman may be obtained by substituting for $F(\theta, \omega)$ from Eq. 67 into Eq. 58. This general result is expressed in integral form. In order to derive more explicit expressions they take $g^2(\alpha, \omega)$ to be of the form $\cos^{2m}(\alpha)$ so that Eq. 67 becomes

$$F(\theta, \omega) = 4m \cos^{2m-1}(\theta), \quad 0 \leq \theta \leq \frac{1}{2}\pi, \\ = 0, \quad \frac{1}{2}\pi < \theta \leq \pi. \quad (69)$$

Substituting from Eq. 69 into 65 and integrating yields¹⁹

$$Q(s, \omega; \pi/2, \xi) = 2^m m! J_m(\omega s/c) / (\omega s/c)^m, \quad (70)$$

which is their result for horizontally separated sensors. Similarly, substituting from Eq. 69 into 64 and letting $x = \cos \theta$ yields

$$Q(s, \omega; 0, \xi) = 2m \int_0^1 x^{2m-1} \exp[i(\omega s/c)x] dx, \quad (71)$$

which is their result for vertically separated sensors. For any value of m , Eq. 71 may be integrated by parts. Explicit expressions are given in most tables of integrals for small values of m .

Liggett and Jacobson¹¹ propose a single parameter family of models of the form:

$$F(\theta, \omega) = 2A \exp(A \cos \theta) / [\exp(A) - 1], \\ = 0, \quad \begin{matrix} 0 \leq \theta \leq \frac{1}{2}\pi, \\ \frac{1}{2}\pi < \theta \leq \pi. \end{matrix} \quad (72)$$

This model has the advantage of not requiring the maximum of $F(\theta, \omega)$ to occur at $\theta = 0$, as did the $4m \cos^{2m-1}(\theta)$ model. It also provides an explicit expression for the cross-spectral density between vertically separated sensors. Substituting from Eq. 72 into Eq. 64

and integrating yields

$$Q(s, \omega; 0, \xi) = \left[\frac{A}{\exp(A) - 1} \right] \left\{ \frac{\exp[i(\omega s/c) + A] - 1}{i(\omega s/c) + A} \right\} \quad (73)$$

for vertically separated sensors.

There are two unattractive features which are shared by the models of Eqs. 69 and 72. The first is that $F(\theta, \omega)$ is identically zero over the lower hemisphere, which is counter to experimental evidence. This is easily overcome by considering Eqs. 69 and 72 as modeling only part of the field. For example, they could be used in a convex combination with isotropic noise as in Eq. 30. The second difficulty is more serious and stems directly from the first. It is that, except for the case $\gamma = 0$ and for Eq. 69 also for $\gamma = \frac{1}{2}\pi$, these models give results for the desired quantities only in the form of difficult integrals. The association of this second difficulty with the first can be seen in Eq. 59 since a finite number of polynomials cannot be identically zero over the interval $\frac{1}{2}\pi \leq \theta \leq \pi$. Thus, the representation of Eq. 59, which is attractive when there are few nonzero terms and leads to explicit results for arbitrary γ , requires an infinite number of terms for the models of Eqs. 69 and 72.

b. Illustration of Spatial Harmonic Approach

The vertical directionality of ambient noise in the deep ocean is dependent on a number of factors including sea state, shipping density, array depth, and frequency. For frequencies below about 100 Hz the noise is attributed to distant shipping and arrives mainly from directions near the horizontal. The detailed structure of the directional density of such low-frequency noise can be expected to vary with depth through a dependence on the sound velocity profile which controls the primary ray paths along which the noise from distant surface ships reaches a particular point in the ocean. At frequencies above about 500 Hz, the noise is attributed to wave action on the ocean surface and more noise arrives from overhead than from directions near the horizontal. In the intermediate frequency range from about 100 to 500 Hz there is a combination of the shipping and surface-wave generated noises. The vertical directionality in this intermediate frequency range is sensitive to sea state and shipping density since these control the relative contributions of the two primary sources of noise. Other effects, such as those due to nearby ships or whales, may cause significant departures from the average conditions described above.

In this section the use of the spatial harmonic approach is illustrated by application to the problem of estimating or modeling the second-order statistical properties of ambient sea noise.

We begin by considering low-frequency noise. Figure 3 presents curves of directional density of ambient

SPATIAL CORRELATION IN NOISE FIELDS

noise at 112 Hz report by Axelrod, Schoomer, and VonWinkle⁵ for Beaufort wind force 4 and 5 based on measurements with a deep, bottom moored vertical line array near Bermuda. They did not report on directions below the horizontal. Also shown in Figure 3 are the curves $10 \log[A_8(\theta)]$ and $10 \log[A_{20}(\theta)]$ where,

$$A_8(\theta) = (1/60) + (59/60) \sin^8 \theta \quad (74)$$

and

$$A_{20}(\theta) = a + (1-a) \sin^{20} \theta \quad (75)$$

for $a=0.001, 0.003, \text{ and } 0.01$. For ease of comparison in the figure, these curves are normalized to have unit values at their maxima. Hence $A_8(\theta)$ and $A_{20}(\theta)$ have to be renormalized to satisfy Eq. 56, which is a power normalization. The function $A_8(\theta)$ has -3-dB points at about ± 23 degrees from the horizontal and contains an isotropic component which is about 18 dB lower than the peak of the directional component. The function $A_{20}(\theta)$ has -3-dB points at about ± 15 degrees from the horizontal and -10-dB points at about ± 26 degrees. Three levels of the isotropic component of $A_{20}(\theta)$, -30 dB , -25 dB , and -20 dB , are shown in Fig. 3. The function $A_{20}(\theta)$ is seen to be within about 1 dB of the curves of Axelrod, Schoomer, and Von Winkle for angles in the range $65^\circ \leq \theta \leq 90^\circ$.

Functions of the form $\sin^{2m}\theta$ are attractive for modeling low-frequency noise because of their peak at the horizontal ($\theta = \frac{1}{2}\pi$) and because they may be represented by a finite series of $m+1$ Legendre polynomials of the form

$$\sin^{2m}\theta = \sum_{n=0}^m a_{2n} P_{2n}(\cos\theta). \quad (76)$$

The coefficients of this series are listed in Table I for $\sin^8\theta$ and $\sin^{20}\theta$.

TABLE I. Coefficients for expansion of $\sin^{2m}\theta$ in Legendre polynomials for $m=4$ and $m=10$.

	$m=4$	$m=10$
a_0	0.40635	0.27026
a_4	-0.73882	-0.58752
a_8	0.46034	0.57107
a_{12}	-0.14776	-0.40735
a_{16}	1.98912×10^{-2}	0.22501
a_{20}	0	-9.68372×10^{-3}
a_{24}	0	3.20229×10^{-3}
a_{28}	0	-7.88417×10^{-3}
a_{32}	0	1.36393×10^{-3}
a_{36}	0	-1.48133×10^{-4}
a_{40}	0	7.60684×10^{-6}

The appropriate normalization so that Eq. 56 is satisfied is achieved by dividing $\sin^{2m}\theta$ by a_0 . For example $(0.40635)^{-1} \sin^8\theta$ satisfies Eq. 56. Similarly, a normalized version of Eq. 75 which maintains the same percent of the isotropic and directional components is

$$F_{20}(\theta) = b + (1-b)(0.27026)^{-1} \sin^{20}(\theta), \quad (77)$$

where

$$b = a[a + (1-a)(0.27026)]^{-1}, \quad (78)$$

while a normalized version of Eq. 74 is

$$F_8(\theta) = 0.04 + (0.96)(0.40635)^{-1} \sin^8\theta. \quad (79)$$

Since $F_8(\theta)$ and $F_{20}(\theta)$ may be written in terms of Legendre polynomials using Eq. 76, it is a simple matter to compute the corresponding cross-spectral density functions using Eq. 60 for any spacing s and angle γ . Figure 4 presents the results of such computations for $\gamma=0^\circ, 30^\circ, 60^\circ, 90^\circ$ when the directional density is described by $F_8(\theta)$ and by $F_{20}(\theta)$. Because $F_8(\theta)$ and $F_{20}(\theta)$ are symmetric about $\theta = \frac{1}{2}\pi$, the corresponding cross-spectral densities are real. Only a single set of curves is shown corresponding to $F_{20}(\theta)$ since the effect of changing a from 0.001 to 0.01 is too small to be seen on the scale of the figure. The effect of narrowing the width of the noise lobe at $\theta = \frac{1}{2}\pi$ by going from $F_8(\theta)$ to $F_{20}(\theta)$ is most evident for vertically separated sensors ($\gamma=0$) where a significant increase in correlation occurs. The effect on horizontal sensors is very small. In the limit of all the noise coming from the horizontal, the cross-spectral density would become unity for vertically separated sensors and $J_0(2\pi s/\lambda)$ for horizontally separated sensors. A comparison of Figs. 4 and 5 shows that the cross-spectral density for horizontally separated sensors when $F(\theta) = F_8(\theta)$ or $F_{20}(\theta)$ is already quite similar to the limit $J_0(2\pi s/\lambda)$ which would occur for $F(\theta) = 2\delta(\theta - \pi/2)$. Thus, certain sensor orientations may be quite insensitive to large changes in the directional characteristics of the field. Hence, care must be exercised in concluding that a particular model is valid on the basis of measurements with sensors in a single orientation.

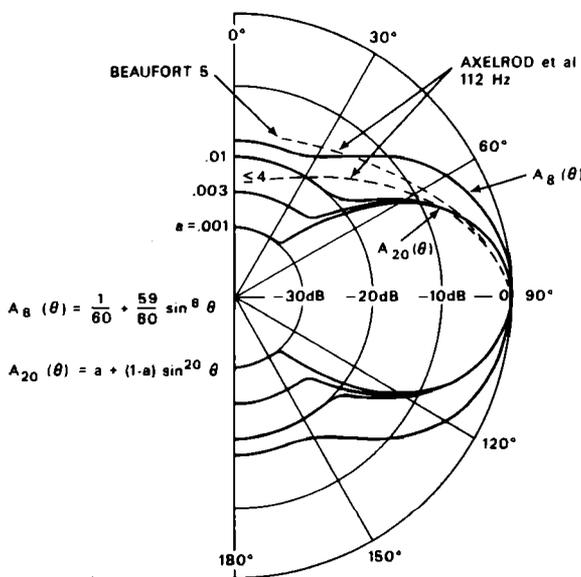


FIG. 3. Comparison of low-frequency noise models with measurements of Axelrod, Schoomer, and Von Winkle.

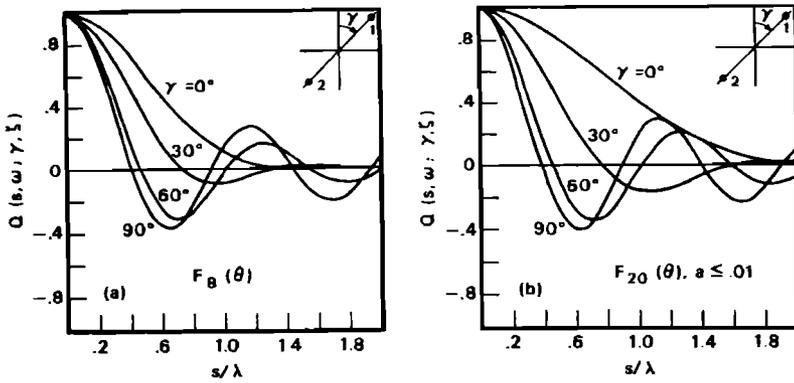


FIG. 4. Cross-spectral density functions for various sensor orientations and spacings computed for $F_8(\theta)$ and $F_{20}(\theta)$ models.

A number of measurements of the normalized spatial correlation at zero time delay have been reported in the literature. If such measurements are made through a bandpass filter of width $\Delta\omega$ and center frequency ω_0 , the measured quantity is related to the cross spectral density function as follows:

$$\rho(s, 0; \gamma, \xi) = (\Delta\omega)^{-1} \int_{\omega_0 - \Delta\omega/2}^{\omega_0 + \Delta\omega/2} \text{Re}[Q(s, \omega; \gamma, \xi)] d\omega. \quad (80)$$

Figure 6 compares low-frequency measurements reported by Arase and Arase⁷ using horizontally spaced bottom-mounted sensors in 3000-ft water depth with the corresponding curve of $\rho(s, 0; \frac{1}{2}\pi, \xi)$ computed for $F(\theta) = F_{20}(\theta)$, $a = 0.001$. The agreement is quite good. A comparison of the curve of Fig. 6 with the $\gamma = 90^\circ$ curve of Fig. 4(b) shows that the use of a finite bandwidth reduces the correlation more as the spacing increases. This effect will be small as long as $s/\lambda_0 \ll \omega_0/\Delta\omega$.

Turning now to the surface wave generated noise at high frequencies, Axelrod, Schoomer, and Von Winkle present curves of directional density at 891, 1122, and 1414 Hz for a number of wind speed conditions. The shape of their curves is quite insensitive to wind speed in this frequency range. A typical example of their results, that at 1122 Hz and Beaufort wind force 5, is given in Fig. 7. Also shown in the figure are the Cron and Sherman model, $F(\theta) \sim \cos\theta$, for $0 \leq \theta \leq \frac{1}{2}\pi$ and a

simple four term harmonic model

$$A_h(\theta) = 0.1 + 0.9[\sin(2\theta)/4 \sin(\theta/2)]^2. \quad (81)$$

This simple model is seen to be in good agreement with the results of Axelrod, Schoomer, and Von Winkle for the region $0 \leq \theta \leq \frac{1}{2}\pi$ which they report. The model gives about 10 dB more noise from $\theta = 0$ than from $\theta = \frac{1}{2}\pi$ and $\theta = \pi$.

Since the function $[\sin(2\theta)/4 \sin(\frac{1}{2}\theta)]^2$ may be expressed as a cosine series using Eq. 32, it is a simple matter to represent it in terms of Legendre polynomials as follows:

$$[\sin(2\theta)/4 \sin(\frac{1}{2}\theta)]^2 = \frac{1}{6} + \frac{3}{10}P_1(\cos\theta) + \frac{1}{3}P_2(\cos\theta) + \frac{1}{8}P_3(\cos\theta). \quad (82)$$

$A_h(\theta)$ may be normalized so that Eq. 56 will be satisfied using the same procedure used to normalize $A_{20}(\theta)$. This yields

$$F_h(\theta) = 0.4 + (0.6)(6)[\sin(2\theta)/4 \sin(\frac{1}{2}\theta)]^2. \quad (83)$$

Figure 8 presents graphs of the real and imaginary parts of $Q(s, \omega; \gamma, \xi)$ when the field is described by $F_h(\theta)$. The imaginary part results from the net flow of noise power downward due to the asymmetry of $F_h(\theta)$ about $\theta = \frac{1}{2}\pi$. The imaginary part for $F_h(\theta)$ does not have as

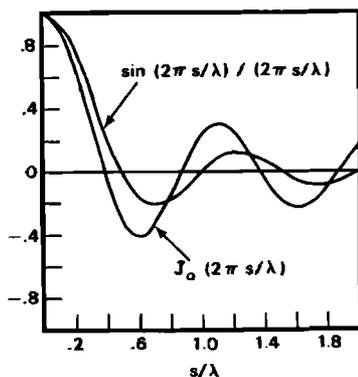


FIG. 5. Cross-spectral density functions for isotropic fields.

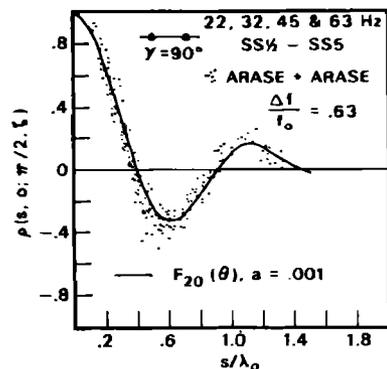
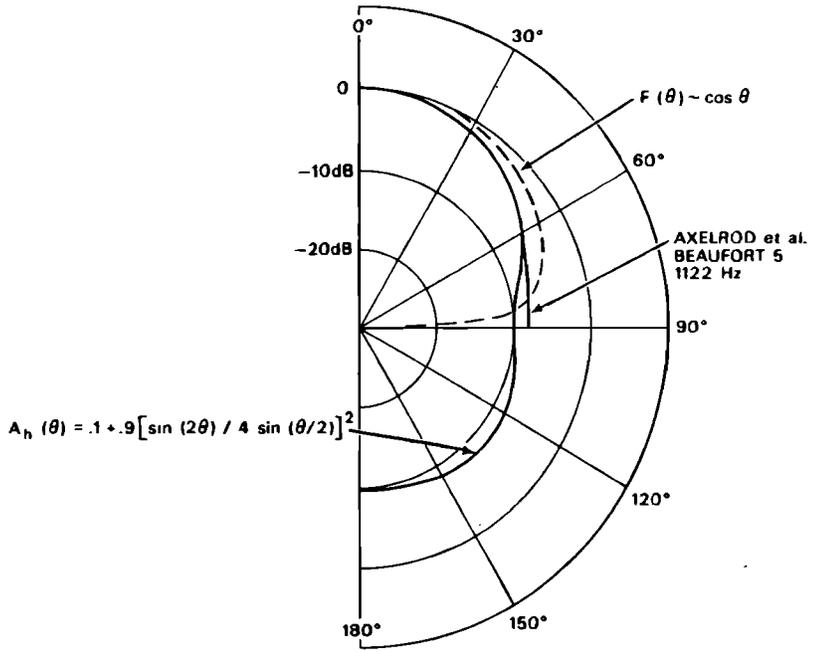


FIG. 6. Comparison of normalized zero time-delay spatial correlation measurements of Arase and Arase with that derived from the $F_{20}(\theta)$ model.

FIG. 7. Comparison of Cron and Sherman model, $A_h(\theta)$ model and high-frequency measurements of Axelrod, Schoomer, and Von Winkle.



high a peak value as it would for the Cron and Sherman model since the asymmetry of the $F_h(\theta)$ model is not as great as that of their model in which $F(\theta)$ is zero for $\frac{1}{2}\pi \leq \theta \leq \pi$.

Figures 9 and 10 compare zero time-delay spatial correlations of vertically separated sensors reported by Cron, Hassell, and Keltonic⁶ with those computed using the $F_h(\theta)$ model. The agreement is quite good.

Having developed a low-frequency model $F_{20}(\theta)$ and a high-frequency model $F_h(\theta)$ which are intuitively reasonable and in apparent agreement with reported data, it is of interest to examine whether a combination of these models will provide reasonable estimates at intermediate frequencies.

The computation of the cross-spectral density function for a mixture of the two models is facilitated by

the use of the following relationship which follows directly from Eq. 43.

Let $Q_m(s, \omega; \gamma, \xi)$ be the cross-spectral density function corresponding to a directional density function $F_m(\theta, \phi, \omega)$. If a directional density function $F(\theta, \phi, \omega)$ is formed by a convex combination of a number of directional density functions as follows:

$$F(\theta, \phi, \omega) = \sum_m \alpha_m F_m(\theta, \phi, \omega), \quad \alpha_m \geq 0, \quad \sum_m \alpha_m = 1, \quad (84)$$

then the corresponding cross-spectral density function is

$$Q(s, \omega; \gamma, \xi) = \sum_m \alpha_m Q_m(s, \omega; \gamma, \xi). \quad (85)$$

A similar relationship holds for the projected wave-number-frequency spectrum.

$$F_h(\theta) = .4 + .38 \left[\frac{\sin(2\theta)}{4 \sin(\theta/2)} \right]^2$$

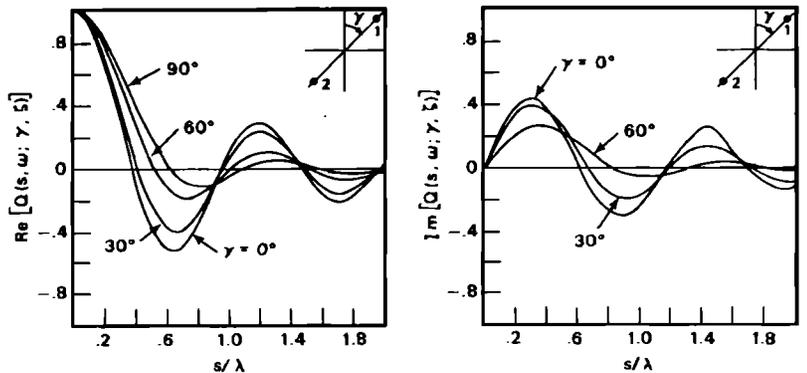


FIG. 8. Real and imaginary parts of cross-spectral density functions for various sensor orientations and spacings based on $F_h(\theta)$ model.

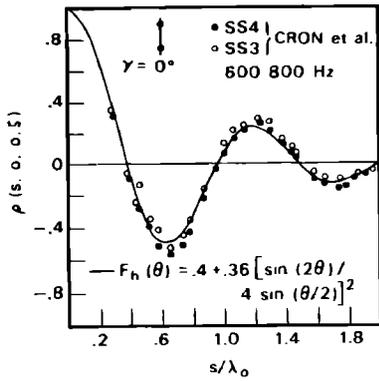


FIG. 9. Comparison of normalized zero time-delay spatial correlation measurements of Cron, Hassell, and Keltonic with that derived from the $F_h(\theta)$ model in the 600-800 Hz band.

The result, Eq. 85, follows from the fact that $F(\theta, \phi, \omega)$ is a linear combination of the $F_m(\theta, \phi, \omega)$. The use of a convex linear combination ($\alpha_m \geq 0, \sum \alpha_m = 1$) is convenient because it guarantees that $F(\theta, \phi, \omega)$ will be nonnegative and the normalization required by Eq. 41 will be retained if each of the component $F_m(\theta, \phi, \omega)$ is nonnegative and normalized. The use of a convex combination is sufficient but not necessary for these conditions to be satisfied. For example, some of the α_m could be negative as long as $\sum \alpha_m = 1$ for normalization and $\sum \alpha_m F_m(\theta, \phi, \omega) \geq 0$ for the nonnegative property.

Thus, if an intermediate frequency noise model is formed using a convex combination of $F_h(\theta)$ and $F_{20}(\theta)$, the cross-spectral density function is easily obtained from those already computed for these components of the model.

Figure 11 illustrates combinations of $F_{20}(\theta)$ with $a = 0.001$ and $F_h(\theta)$ of form

$$F(\theta) = B F_h(\theta) + (1-B) F_{20}(\theta), \quad (86)$$

where the values $B = 0.01, 0.1, 0.5$ and 0.8 are shown. Figure 12 presents zero time-delay spatial correlations at 250 Hz reported by Arase and Arase⁷ for vertically

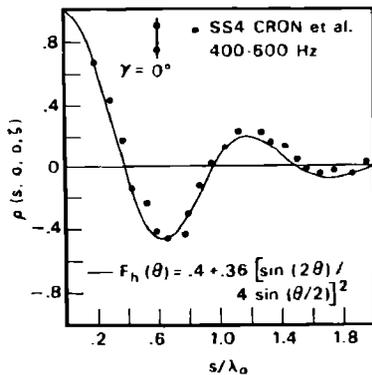


FIG. 10. Comparison of normalized zero time-delay spatial correlation measurements of Cron, Hassell, and Keltonic with that derived from the $F_h(\theta)$ model in the 400-600 Hz band.

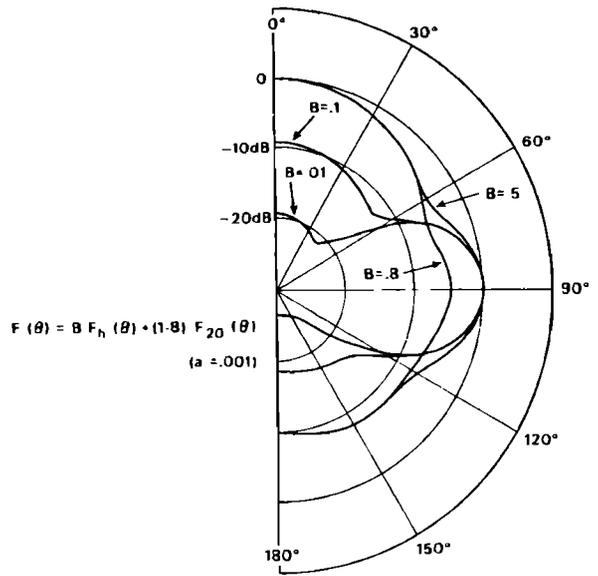


FIG. 11. Directional density functions obtained by combining the high-frequency $F_h(\theta)$ and the low-frequency $F_{20}(\theta)$ models.

separated hydrophones in a sea state 5. Also shown is the curve based on the model of Eq. 86 with $B = 0.8$ so that there is 6 dB more surface noise than shipping noise. Again the agreement is quite good. Average ambient noise level curves such as those given by Urick²¹ indicate that for sea state 5 at 250 Hz, 6 dB more surface noise than shipping noise would be consistent with a moderate shipping density. Thus the value $B = 0.8$ is intuitively reasonable.

The simple models $F_{20}(\theta)$ and $F_h(\theta)$ are consistent with a considerable body of measured data. They may be used to estimate performance of arrays in a variety of noise fields. They also facilitate the examination of the sensitivity of experimental setups with particular sensor orientations to particular characteristics of noise fields. They cannot be expected to fit all situations, nor

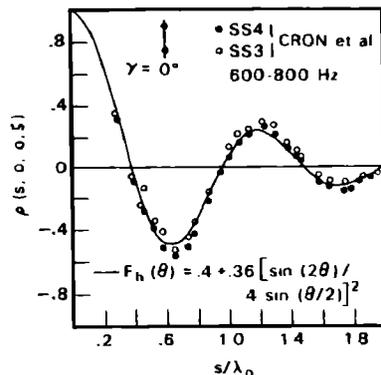


FIG. 12. Comparison of normalized zero time-delay spatial correlation measurements of Arase and Arase at 250 Hz with that derived using a mixture of the $F_h(\theta)$ and $F_{20}(\theta)$ models with 80% of the noise power attributed to $F_h(\theta)$.

is it claimed that they are the best models for any particular situation. Rather they have been presented to illustrate a powerful approach and to present some initial results which may be useful until better models are developed.

IV. CONCLUSIONS

By expanding the directional density function in spatial harmonics series representations rather than the usual integral representations are obtained for the cross-spectral density function and the projected wavenumber-frequency spectrum. For each term in these series, the dependence on the position coordinates of the sensors is factored into components, each of which depends on a single coordinate. This property of the series expansion facilitates the calculation of cross-spectral densities for various sensor orientations. This can be a significant advantage over the integral representation when all pairs of sensors in a large array must be considered. The series representations are particularly attractive when the directional density function may be adequately represented by a relatively small number of harmonics so that the series terminate after a correspondingly small number of terms. This will be the case when the directional density function is fairly smooth with gradual changes.

The use of the spatial harmonic approach has been illustrated by application to the vertical distribution of ambient noise. Two simple models have been presented—a high-frequency model with four terms in the series and a low-frequency model with eleven terms in the series. It has been shown that these models, alone or in combination, are in good agreement with a large body of measured data. An important advantage of these models over others which have been suggested is the ease of obtaining results for arbitrary orientations of the sensors.

It is a simple matter to develop better models as information about the structure of noise fields becomes more complete. Such models can be built up from a convex combination of simpler models. This approach has the advantage that the cross-spectral density func-

tion and the projected wavenumber-frequency spectrum will be corresponding convex combinations of those for the component models.

*This work was done while the author was a visiting Research Associate under the sponsorship of the U. S. Navy Professional Development Program and was supported in part by the Office of Naval Research.

†*Dedication:* To my parents with love and gratitude on the occasion of their fortieth wedding anniversary, February 12, 1974.

‡*Current address:* New London Laboratory, Naval Underwater Systems Center, New London, Connecticut 06320

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