

# Heat Transfer in the Earth – Part III

## See Chapter 4 of Lowrie

### Halfspace cooling – formal solution

Let us proceed from the dimensional analysis to obtain the formal solution for the temperature distribution in a cooling halfspace. We know by dimensional analysis that a halfspace initially at temperature  $T_0$  and with the surface suddenly raised to a temperature  $T_s$  will suck in heat according to

$$q \approx K \frac{(T_s - T_0)}{L} \approx K \frac{(T_s - T_0)}{\sqrt{\kappa t}}$$

and thus we want to find a solution for  $T(z, t)$  which when differentiated with respect to  $z$  and multiplied by  $K$  yields a solution like the one above at  $z = 0$ .

Let us return to the one-dimensional diffusion equation

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\kappa} \frac{\partial T}{\partial t} = 0$$

Because this partial differential equation is second-order in  $z$  and first-order in  $t$ , we need to specify two boundary conditions (in  $z$ ) and an initial condition at  $t = 0$ .

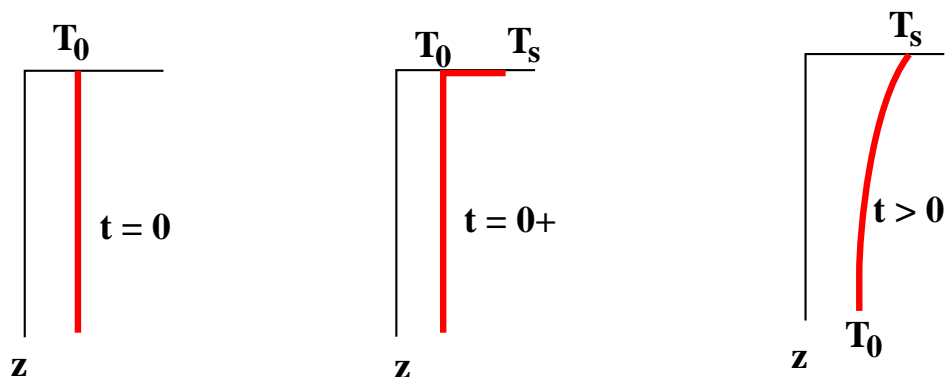


Figure 9

The conditions are

$$\begin{aligned}T &= T_0 \text{ at } t = 0, z \geq 0 \\T &= T_s \text{ at } t > 0, z = 0 \\T &\rightarrow T_0 \text{ at } t > 0, z \rightarrow \infty\end{aligned}$$

The solution can be shown to be

$$T(z, t) = T_0 + (T_s - T_0) \operatorname{erfc} \left[ \frac{z}{\sqrt{4\kappa t}} \right]$$

where  $\operatorname{erfc}(x)$  is a *complimentary error function*

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

and the *error function* is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$$

The error function is an “S” shaped curve with properties

$$\operatorname{erf}(0) = 0; \quad \operatorname{erf}(\infty) = 1$$

This error function solution is the most fundamental type of solution in geophysics for transient cooling or heating of a conductive body. We will use this solution shortly to examine the cooling of the oceanic lithosphere. Notice that the solution says that when the surface temperature is suddenly increased from  $T_0$  to  $T_s$ , this higher temperature diffuses downward into the ground with time.

The first derivative of an error function is given by

$$\frac{d\operatorname{erf}(x)}{dx} = \frac{2}{\sqrt{\pi}} \exp(-x^2)$$

so heat flux must be given by

$$q(z,t) = K(T_s - T_0) \frac{2}{\sqrt{\pi}} \exp\left(\frac{-z^2}{4\kappa t}\right) \frac{1}{\sqrt{4\kappa t}}$$

and

$$q(0,t) = \frac{1}{\sqrt{\pi}} K \frac{(T_s - T_0)}{\sqrt{\kappa t}}$$

Comparison to our earlier solution shows that the dimensional analysis result is only off by a factor of

$$\frac{1}{\sqrt{\pi}}$$

Not bad!! Notice that the heat flow starts off very high and then declines with one over the square root of time.

## Cooling Oceanic Lithosphere

### *Horizontal advection*

A classic problem is to apply the solution of the diffusion equation to the cooling, subsiding oceanic lithosphere. The 1-D space problem can turn into a 2-D space problem by assuming heat is advected horizontally at velocity  $V_x = V$ . So instead of waiting a time  $t$  to observe cooling, simply step out a distance  $x = Vt$  to observe it. This obviously has application to the cooling oceanic lithosphere spreading away from oceanic ridges.

Since

$$dx = V dt$$

then the diffusion equation becomes

$$\frac{\partial^2 T}{\partial z^2} - \frac{V}{\kappa} \frac{\partial T}{\partial x} = 0$$

The initial condition is

$$T(x=0, z) = T_m$$

Note that the initial condition is now at  $x = 0$  instead of  $t = 0$ . and the boundary conditions are

$$T(x, z = 0) = T_0$$

$$T(x, z = \infty) = T_m$$

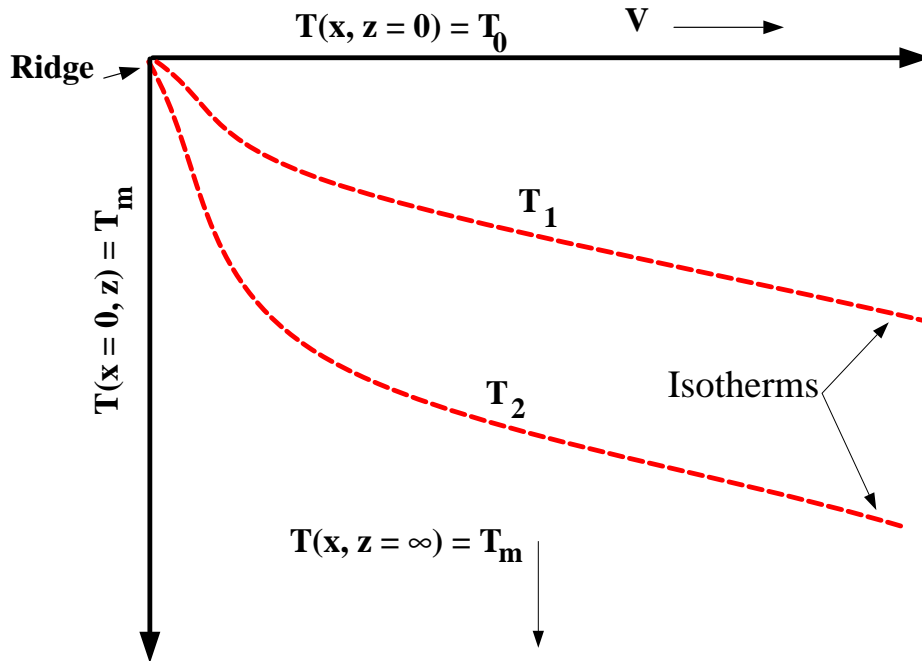


Figure 10

$T_m$  is obviously meant to represent the hot asthenosphere beneath the lithosphere. This hot material reaches the surface at the ridge axis, where new oceanic crust is formed from cooling basaltic magma.

By inspection of our general error function solution from above, the result here is obviously

$$T(x, z) = T_0 + (T_m - T_0) \operatorname{erf} \left[ \frac{z}{\sqrt{4\kappa x/V}} \right]$$

because this form satisfies the initial and boundary conditions.

Approximate isotherms of this solution are shown in Figure 10 above. Note that the heat flux at the surface is given by

$$q(x,0) = \frac{1}{\sqrt{\pi}} K \frac{(T_m - T_0)}{\sqrt{\kappa x/V}}$$

and falls off as the square root of distance. **This diffusive cooling of the oceanic lithosphere a major way in which the Earth loses its internal heat!!!**

### Isostasy

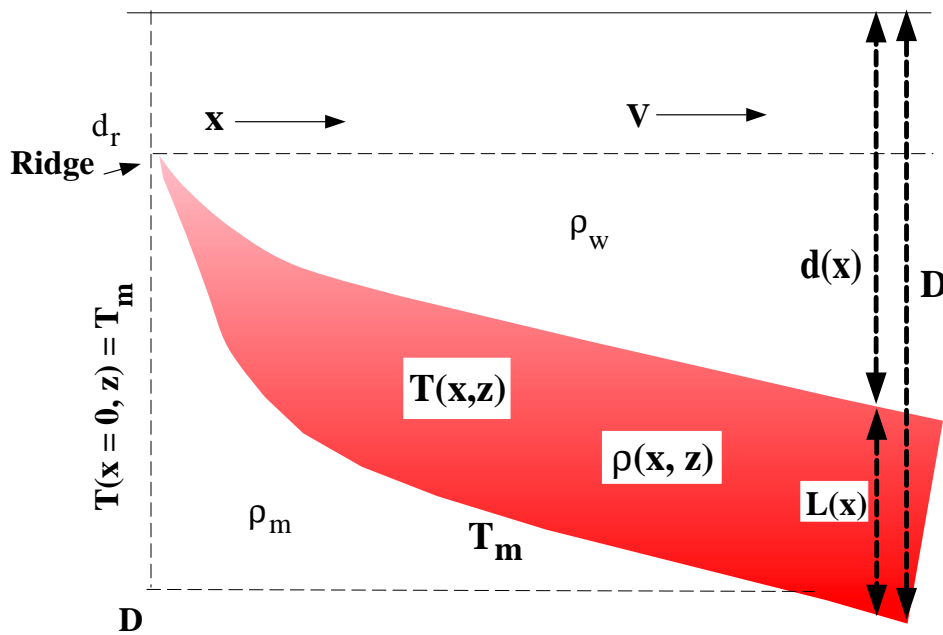


Figure 11

The oceanic lithosphere adjusts isostatically as it cools. Thus we can calculate the depth of the ocean floor as a function of distance from the ridge axis. The principle of making the isostatic calculation is to note that the weight of a column to a depth  $D$  at the base of the lithosphere at any distance  $x$  from the ridge axis must be equal to the weight of the sublithosphere (a.k.a. asthenosphere) at depth  $D$  beneath the ridge axis. Putting  $z = 0$  at the ocean surface, then isostatic balance

requires

$$g_0 \int_0^{d_r} \rho_w dz + g_0 \int_{d_r}^D \rho_m dz = g_0 \int_0^{d(x)} \rho_w dz + g_0 \int_{d(x)}^{d(x)+L(x)} \rho(x, z) dz$$

Integrating where density is constant and rearranging yields

$$[d(x) - d_r](\rho_m - \rho_w) = \int_0^{L(x)} [\rho(x, z) - \rho_m] dz$$

The density in the lithosphere can be calculated knowing the temperature distribution and using the coefficient of thermal expansion,  $\alpha$ :

$$\rho[T(x, z)] = \rho_m \{1 - \alpha [T(x, z) - T_m]\}$$

Using our temperature solution (derived above) for the cooling oceanic lithosphere

$$T(x, z) = T_0 + (T_m - T_0) \operatorname{erf} \left[ \frac{z}{\sqrt{4\kappa x/V}} \right]$$

then

$$[d(x) - d_r](\rho_m - \rho_w) = \rho_m \alpha T_m \int_0^{L(x)} \operatorname{erfc} \left[ \frac{z}{\sqrt{4\kappa x/V}} \right] dz$$

where, again,

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

The integral above can be calculated numerically, but we can let

$$L(x) \rightarrow \infty$$

and introduce a 5% error. The integral from 0 to infinity of  $\operatorname{erfc}(x)$  is just

$$\int_0^\infty \operatorname{erfc}(x) dx = \frac{1}{\sqrt{\pi}}$$

so with a change of variables our solution becomes

$$[d(x) - d_r](\rho_m - \rho_w) = 2\rho_m \alpha T_m \sqrt{\frac{\kappa x}{\pi V}}$$

and rearranging

$$d(x) = d_r + \frac{2\rho_m \alpha T_m}{(\rho_m - \rho_w)} \sqrt{\frac{\kappa x}{\pi V}}$$

We see that depth to the ocean floor deepens as the square root of the distance  $x$  from the ridge axis. Is this true? Yes, pretty much out to a seafloor age of about 100 Ma. Then the sea floor tends to flatten out, as does the heat flow, which is predicted to decrease as  $1/\sqrt{x}$ . **Why is this? And what are the effects of changing  $V$  and  $\kappa$ ? Can we see the effect of different values of  $V$  in ocean floor topography?**