Heat Transfer in the Earth – Part II

See Chapter 4 of Lowrie

Heat Sources

Background

What gives rise to heat sources within a planet? Here are some possible sources:

1. Original heat of accretion.

2. Conversion of gravitational potential energy (e.g., core formation)

3. Exothermic phase or compositional changes.

3. Decay of radiogenic isotopes.

4. Mechanical heat dissipation

The original heat of accretion arises from the conversion of gravitational potential energy associated with the infall of material to form the planet. Terrestrial--sized planets have lost essentially all record of their original heat, whereas the larger giant planets have not. The conversion of gravitational potential energy when the Earth's core was formed was sufficient to form the core in a molten state. The core is still cooling and releasing heat. The formation of the solid inner core at the expense of the molten outer core also produces heat and helps maintain the geodynamo. Decay of short-lived isotopes (e.g., $^{26}Al \rightarrow {}^{26}Mg$) may be responsible for differentiating meteorite parent bodies. By far the largest source of heat in the terrestrial planets is the decay of long-lived radiogenic isotopes.

Decay of long-lived isotopes

Unstable radiogenic isotopes decay according to

$$\frac{dn}{dt} = -\lambda n$$

or

$$n(t) = n_0 e^{-\lambda t}$$

This radioactive process is manifested in the emission of particles or electromagnetic radiation. The two major types of emissions are α particles (charged helium nucleus) and β particles (electrons). Both types of particles are brought to rest in a rock within a few mm of where they are emitted. When this happens, their kinetic energy is converted to heat, which can be precisely measured in the laboratory.

We can measure the heat flow at the surface of the Earth and get some idea of the heat production in the mantle. Under the assumption of steady state (time derivatives are zero, and zero velocity), the divergence of the heat flux vector is equal to the heat production rate, as stated above.

$$\nabla \cdot \mathbf{q} = Q$$

One dimensionally,

$$\frac{dq_z}{dz} = Q$$

But if we are considering the whole Earth, we would do better in spherical coordinates, where the one dimension is the radius *r* measured from the center of the Earth. So our equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 q_r\right) = Q$$

Assuming Q is constant and integrating

$$r^2 q_r = \frac{r^3}{3} Q + C$$

and $q_r(0) = 0$, so $C = 0$. Letting r = R_0, the radius of the Earth, and multiplying both sides by 4π shows that the total heat coming out of the Earth

$$4\pi R_0^2 q_r = \text{total heat produced} = \tfrac{4}{3}\pi R_0^3 Q$$

Big deal: the volume production of heat is equal to the amount of heat escaping across the surface. If $\overline{q}_r$ is the Earth's average heat flux, then

$$\overline{q}_r = \frac{R_0}{3} Q$$

How useful is this for estimating the heat source content of the mantle?

1. It neglects heat coming out of the core, which is thought to generate thermal plumes and create hotspots -- perhaps a 10% effect. How could you measure this effect?

2. It neglects radiogenic isotopes that have been fractionated into the crust.

3. The Earth is not in steady state, but is cooling. What we measure today is an integral effect over several Gyr, and heat production was higher in the past.

We define *Q*' as the heat production per unit mass by dividing *Q* by density. Here are the steps to obtain *Q*' for the mantle:

1. Take average heat flow (~82 mW m^{-2}) and use the relationship above to estimate 8.88×10^{-12} W kg^{-1}.

2. Multiply by 0.9 to correct for core contribution = 7.99×10^{-12} W kg^{-1}.

3. Take mean continental heat flow of 56.5 mW m^{-2} and estimate that 23 mW m^{-2} comes from the crust (How would you figure this out?). Use fraction of continental crust on planet to find correction factor of 0.87 to yield 6.95×10^{-12} W kg^{-1} .

4. Do a convection calculation and discover about 20% of heat comes from cooling of the Earth. So $Q = 5.56 \times 10^{-12}$ W kg^{-1}.

We can relate this result to independent estimates of the heat source concentration of the mantle and the results compare fairly well. (Where might the independent estimates come from?). What could we learn about the Earth if we really had a good estimate of heat source concentration?

Mantle heat production arises from ^{238}U, ^{235}U, ^{232}Th, and ^{40}K. The contribution of each isotope individually depends on the its intrinsic heat generation and initial abundance and half-life (or, alternatively, its present abundance). The table below is instructive: The total heat production has dropped by about a factor of three over that last 4 Gyr. The isotopes ^{238}U and ^{232}Th dominate heat production today.

Isotope	Heat Production* W kg^{-1} x 10^5	Half-life Gyr	Concentration* kg kg^{-1} x 10^9
^{238}U	9.37	4.47	25.5
^{235}U	56.9	0.704	0.185
^{232}Th	2.69	14.0	103
^{40}K	2.79	1.25	32.9

*Heat production based on present mean mantle concentrations

Steady State Solutions

Simplest results

The simplest configuration we can imagine is a *steady state* (time derivative = zero), static (**V** = 0) medium with no heat sources. The temperature then satisfies(yes!!) LaPlace's equation

$$\nabla^2 T = 0$$

This tells us, by the way, that in steady-state, temperature is a potential field, and the corresponding "force" field is heat flux **q** (and $\mathbf{q} = -K\nabla T$). With temperature variations in the *z*-direction only (e.g., a layered halfspace), we obtain the even simpler

$$\frac{d^2T}{dz^2} = 0$$

and we can write down the solution immediately as

$$T(z) = Az + B$$

which requires 2 boundary conditions for solution. One boundary condition is temperature specified at the surface:

$$T(0) = T_0 = A \cdot 0 + B$$
$$\Rightarrow B = T_0$$
$$T(z) = Az + T_0$$

For the second boundary condition, assume we are interested in temperature in a layer of thickness *L*. We specify the heat flux at the base of the layer as q_m. Then

$$T(z) = Az + T_0$$
$$q(z) = -K\frac{dT(z)}{dz} == -KA = q_m$$
$$\therefore$$
$$T(z) = -\frac{q_m}{K}z + T_0$$

Now put a constant heat source in the layer:

$$\frac{d^2T}{dz^2} = -\frac{\rho Q'}{K}$$

The solution is

$$T(z) = T_0 - \frac{q_m}{K}z - \frac{\rho Q'}{2K}z^2$$

a quadratic function in *z*. This is a good approximation for the temperature distribution in the crust. But we need to know the heat flux coming into the base of the crust (q_m) and the heat source concentration in the crust (Q')

Continental Heat Flow

Assume that heat production in the crust declines exponentially with depth according to

$$Q'(z) = Q_0' e^{-z/h_f}$$

where h_f is the e-folding depth for the decrease in heat production. Why would this be a good model? We consider two boundary conditions. At the surface $T = T_0$ and at great depth the heat flux approaches that coming from the mantle, q_m. The solution is

$$T(z) = \left(T_0 + \frac{h_f^2 \rho Q_0'}{K}\right) - \frac{q_m}{K} z + \frac{h_f^2 \rho Q_0'}{K} e^{-z/h_f}$$

At the surface

$$q_s = q(0) = -K\frac{dT}{dz} = q_m + h_f \rho Q_0'$$

So we see that heat flow has a linear dependence on heat production as in

$$q_s = a + bQ_0'$$

and this is borne out by measurements of heat flow over granite plutons and the corresponding measurements of heat production in samples of surface rocks. The quantity q_m is termed the *reduced heat flow*, and it is estimated by stripping of the effects of the crust as in the above equation. This tells us the heat coming from the mantle beneath the continents.

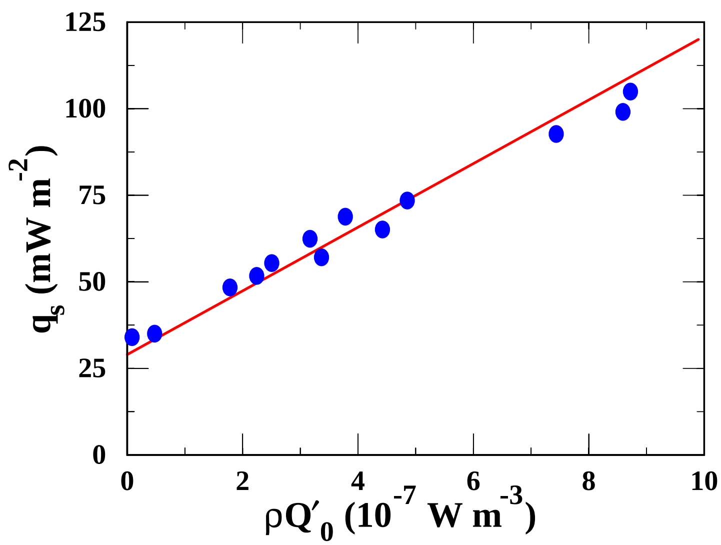


Figure 5. Linear dependence of surface heat flux in the eastern U.S. on the radioactive heat content of surface rocks. The blue dots are measured values, and the red line is a least-squares fit of a straight line to the data. The mantle heat flow is obtained from the value at $\rho Q_0' = 0$.

Time-dependent Solutions to the Heat Equation

We turn to *time-dependent* solutions of the heat equation.

Dimensional analysis

Consider the one dimensional diffusion equation with zero velocity and no heat sources:

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\kappa}\frac{\partial T}{\partial t} = 0$$

Writing out the heat equation in dimensional form:

$$\frac{T}{L^2} = \frac{T}{\kappa\tau}$$

where L is a length scale and τ is a time scale. So dimensionally,

$$L^2 = \kappa\tau$$

and this is a handy "back-of-the-envelope" relationship for cooling or heating a body of characteristic dimension L. For rocks the "canonical" value of κ is 10^{-6} m^2 s^{-1}. Since there are about π x 10^7 seconds in a year, the above equation can be rewritten as

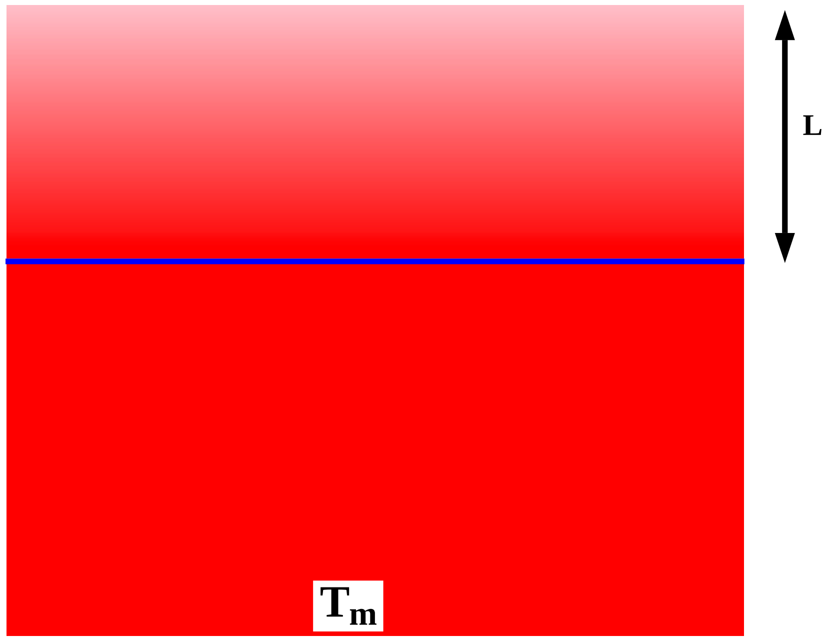
$$L^2 = \pi \times 10^{-5}\tau$$

where L is in km and τ is in years. So, for example, a 100-km-thick layer (subduction slab, batholith, etc.) would cool (or heat) about 1/e in 3.2x10^8 years. This result is good to about half an order of magnitude.

This dimensional analysis result can be determined in a more intuitive way. Consider an infinite halfspace at some initial temperature, T_m (e.g., a large cooling batholith). After a period of time, τ, cooling (which proceeds from the "cool" surface of the Earth inwards, by diffusion) has proceeded to some depth, L, and dimensionally

$$q \approx K\frac{T_m}{L}$$

Batholith cools from top down. Cooling gives off a heat flux q. After a time τ, cooling has proceeded to a depth L.



$$\mathbf{q \approx K\, T_m/L}$$

Figure 6.

The total amount of heat energy (in Joules) removed from the halfspace after time τ must be

$$q_T \approx q\tau \approx \frac{\tau K T_m}{L}$$

The amount of heat energy removed due to cooling must *also* be

$$q_T = \rho C_p T_m L$$

Equating the two expressions,

$$\tau = L^2 \frac{\rho C_p}{K} = \frac{L^2}{\kappa}$$

which agrees with the previous result

Periodic solutions to the heat equation

One of the simplest solution to the heat equations assumes that time variations are periodic and the spatial dependence varies with depth only:

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\kappa}\frac{\partial T}{\partial t} = 0$$

and

$$T(z,t) = \sum_{n=0}^{\infty}\left[A(z)\cos(\omega_n t) + B(z)\sin(\omega_n t)\right]$$

where radial frequency is related to angular frequency and period by

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Why can the temperature be solution be written in such a way? However, we do not need both trigonometric terms in the solution. We can get away with one term (say, the sine function) and a *phase shift*. The phase shift just shifts the zero location of the sine function. At the surface, $z = 0$, the temperature is given by

$$T(0,t) = T_0 + \sum_{n=1}^{\infty}\Delta T_n \sin\left[\omega_n (t - t_0)\right]$$

where the sine wave passes through zero at $t = t_0$. With this boundary condition, the overall solution to the diffusion equation is

$$T(z,t) = T_0 + \sum_{n=1}^{\infty}\Delta T_n \exp\left(-\sqrt{\frac{\omega_n}{2\kappa}}z\right)\sin\left(\omega_n t - \sqrt{\frac{\omega_n}{2\kappa}}z\right)$$

where we have for convenience set $t_0 = 0$. Clearly at depth, the temperature is attenuated and phase-shifted. The higher the frequency (shorter the period), the more rapidly will a surface temperature variation attenuate with depth. This equation has application to heating of the subsurface by variations in surface temperature. The two strongest *periodicities* are obviously the daily wave, which is important to about 25 cm depth, and the annual wave which is important to about 1 meter depth. So we need only to keep two terms in (n values) in the series solution above. **What are some other important periodicities?**

The two figures (Figures 7 and 8) following show the surface temperature and the temperature at 75 cm depth over a period of a year at site near Taos, NM. The surface temperature is approximately the daily average, so the high frequency variations are temperature changes over several days, probably associated with weather fronts. The record at 75 cm depth shows that most of the high frequency fluctuations have been damped out in accordance with the equation above. The second figure shows sine-wave fits to the annual wave in the data sets, clearly indicating attenuation and phase shift of the annual wave in the subsurface. ***Geophysics works!!***

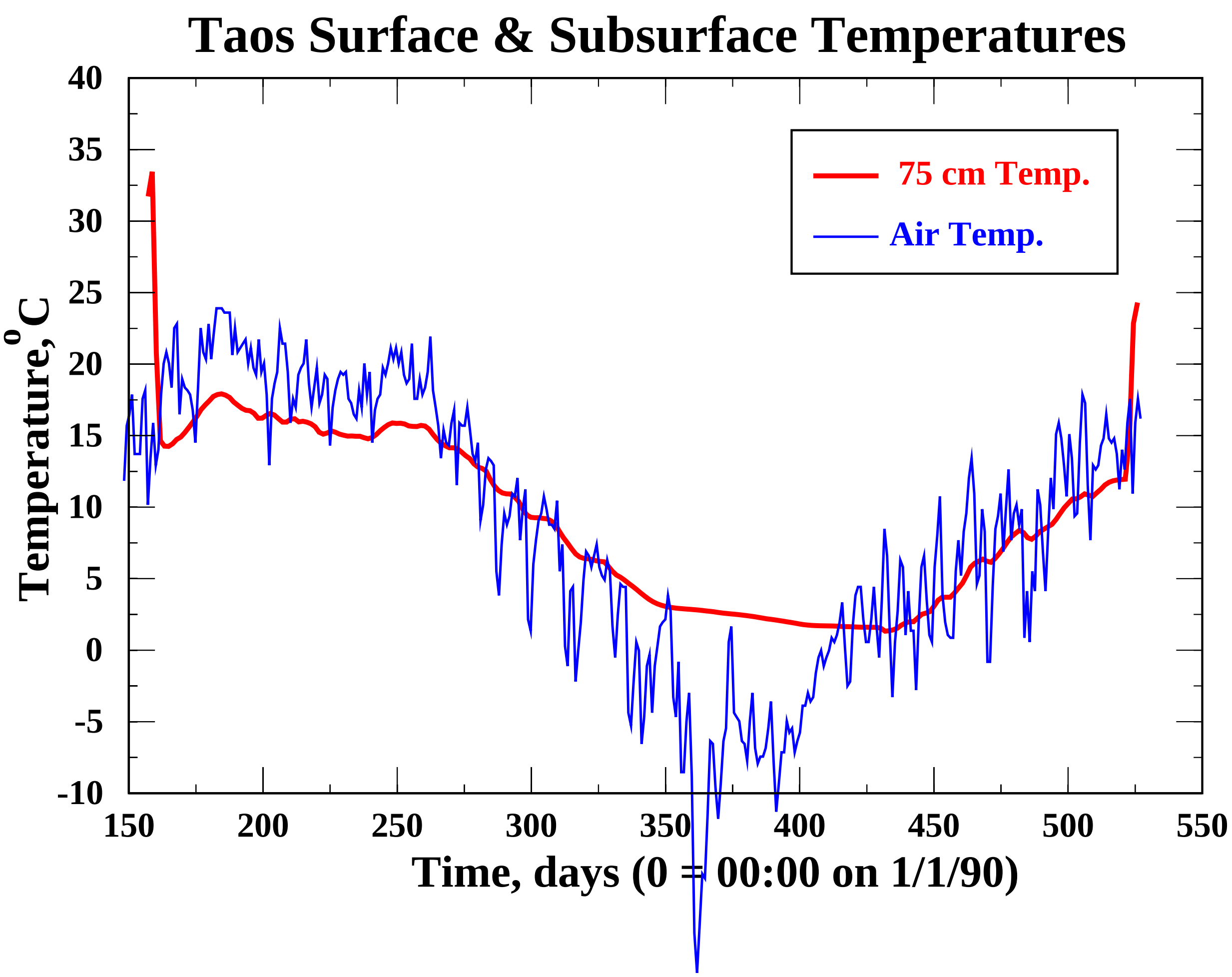


Figure 7

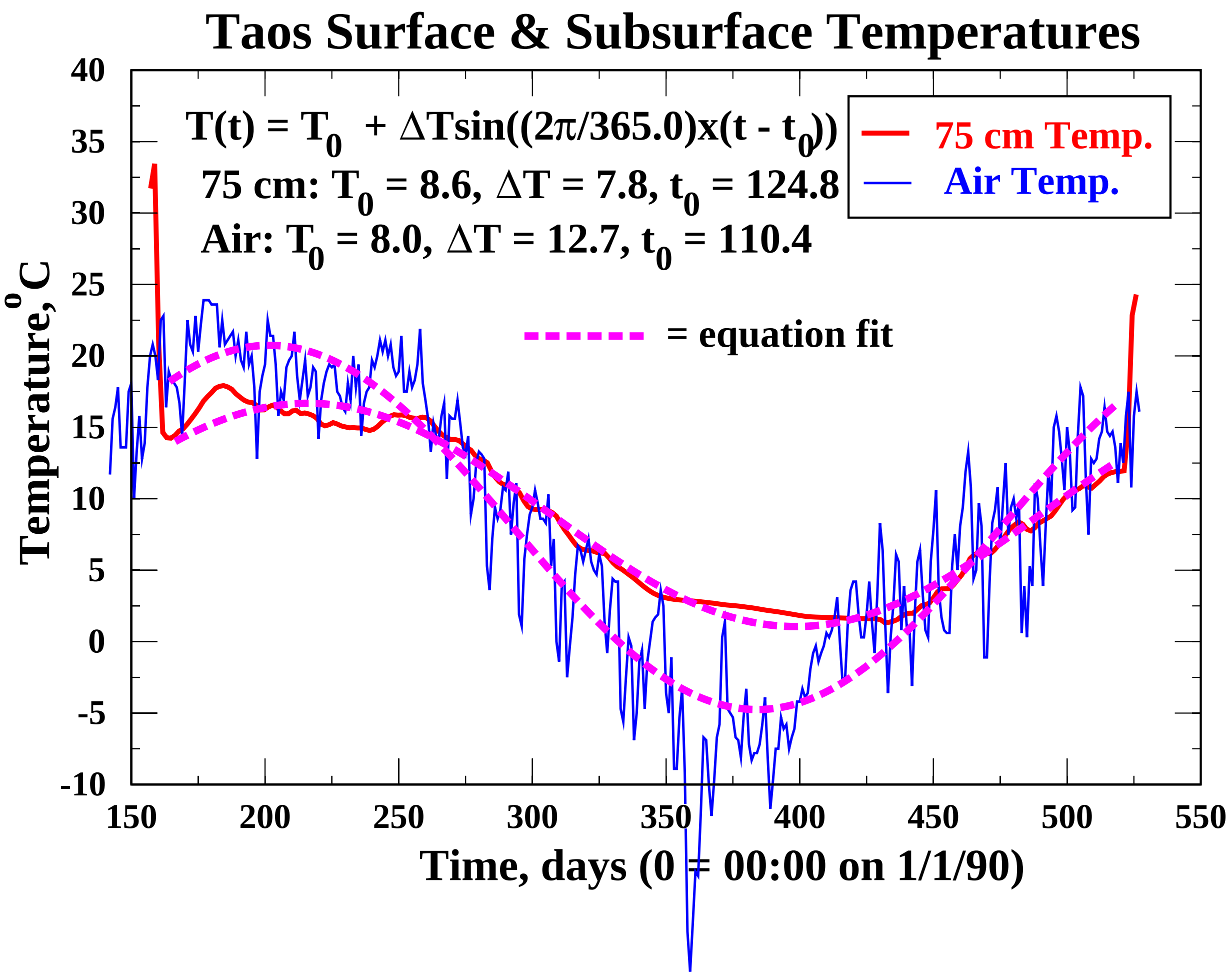

Taos Surface & Subsurface Temperatures
T(t) = T0 + ΔTsin((2π/365.0)x(t - t0))
75 cm: T0 = 8.6, ΔT = 7.8, t0 = 124.8
Air: T0 = 8.0, ΔT = 12.7, t0 = 110.4
75 cm Temp.
Air Temp.
= equation fit
Temperature, oC
Time, days (0 = 00:00 on 1/1/90)
40
35
30
25
20
15
10
5
0
-5
-10
150
200
250
300
350
400
450
500
550

Figure 8