# Heat Transfer in the Earth – Part I

# See Chapter 4 of Lowrie

## 1. <u>Conservation of Energy</u>

***Energy can neither be created nor destroyed***.

- What does this statement mean in physical terms?
    - The form of energy may change, but the amount of energy in the Universe is a fixed quantity (Of course, mass is just another form of energy, as in $E = mC^2$.).
- Examples
    - **<u>Gas Stove</u>** : Something known as the chemical potential of the natural gas is converted to heat energy through the action of burning. This energy is then transferred to the air and to the pot on the stove. The pot then heats up and heats up the water in the pot, etc.
    - etc...

## 2. <u>Heat Transfer</u>

- The first question is: what really is *heat*?
    - Kinetic energy is related to motion, potential energy is related to the ability to do work, and heat is just the energy that we relate to temperature.
- How is heat transferred?
    - **Three modes of heat transfer**
        - Conduction
        - Convection
        - Radiation

- **Conduction**
  - What is conduction and how does it work?
  - Conduction is the transfer of energy through the collisions and vibrations of molecules and atoms.
    - The temperature of molecules are related to their velocity.
    - Conduction through a mineral is a good way to illustrate vibration of atoms.
  - For example, look at a cubic mineral.

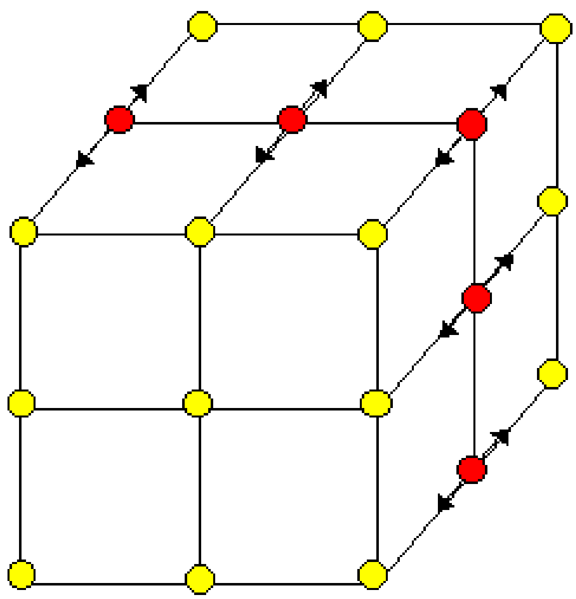

**Figure 1**

- How does all of this relate to something that we can use?
- The best way to start think of conduction is to visualize a simple slab.

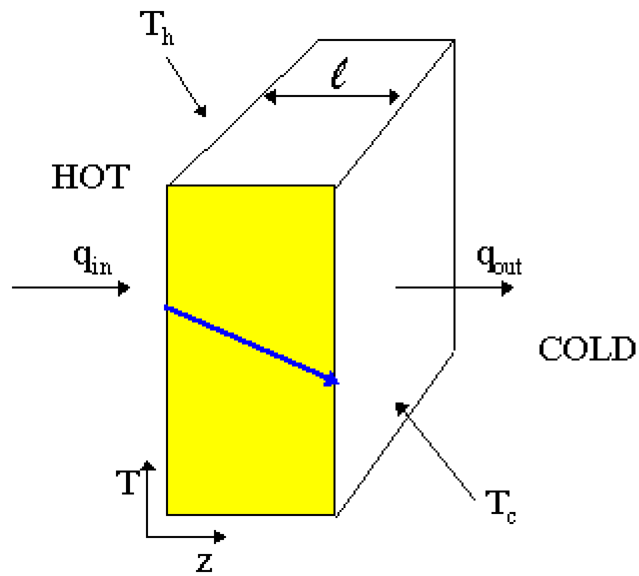


**Figure 2**

- This slab can be anything from wood to granite to home insulation to metal or the **lithosphere**.
- What is the temperature gradient across the slab?
  - The change in temperature over the distance *l* is given by.

$$\text{Temperature Gradient} = \frac{T_h - T_c}{l} = \frac{\Delta T}{l} \rightarrow \frac{dT}{dz}$$

- **q** is the heat flux
  - The flow of heat energy (Joules, J) across a unit area per unit time (or J/s-m$^2$ = W/m$^2$.)
- The heat flux is proportional to the temperature gradient

$$q \sim \frac{dT}{dz}$$

- How do we make the heat flux **EQUAL** to something? We need a constant of proportionality:

$$q = -K\frac{dT}{dz}$$

- $K$ is the thermal conductivity (units of $Wm^{-1}K^{-1}$). This relationship holds in a conductive medium.
- The negative sign indicates that the heat is flowing from the hot to cold.
- This equation is known as Fourier's Law of Heat Conduction.
- You will be pleased to know, and it is intuitively reasonable, that heat will always flow in the direction of the maximum change in temperature, or the highest temperature gradient. You will then tell me that a 3-diemsional generalization of Fourier's Law must be given by

$$\mathbf{q} = -K\nabla T$$

and you are right!

- **How is the Energy Balanced?**
  - What are the energy components of this system?
  - Energy balance is similar to mass balance for a system.
  - **Example**
    - Think of a sediment transport in a river. How would you balance the *rate* of mass transport of the sediment?

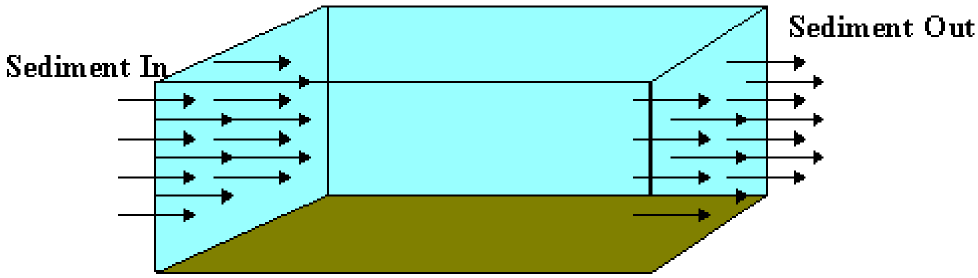


Figure 3

Rate of Sediment Out = Rate of Sediment In + Rate of Sediment Deposition

- Now back to the Energy Balance….. similar to above

Rate of Heat Energy Out = Rate of Heat Energy In + Rate of Change of Heat Content

or

(Rate of Heat Energy Out - Rate of Heat Energy In) = Rate of Change of Heat Content

- This assumes no heat sources or sinks in the system.
- **What would be a heat source in an Earth material?**

- **What is Heat Content?**
    - It can be expressed as

$$\rho C_p \Delta T$$

- $\rho$ is the density
- $\boldsymbol{C_p}$ is the specific heat with units of J $kg^{-1}$ $K^{-1}$. It is the amount of energy required to raise 1 kg of material by 1 degree Kelvin.

$$C_p = \frac{dE}{dT}$$

- $E$ is the Heat Energy and the subscript $p$ denotes that the derivative is taken with pressure kept constant.
    - Keep in mind the constant pressure part for the future: remember, the interior of the earth is not at a constant pressure as a function of depth!
    - As mentioned, specific heat is the energy required to raise 1 kg of material by 1 degree K. It is a property of the material.
    - The rate of change in heat content over a time interval $\Delta t$ is

$$\rho C_p \frac{\Delta T}{\Delta t} \rightarrow \rho C_p \frac{\partial T}{\partial t}$$

    where we use a partial derivative because $T$ depends on space as well as time. This is the right hand side of our energy rate statement.

- More interesting to us is the way heat content changes with temperature. **Why?**
    - An increase in heat content with time means that **<u>less</u>** heat flows out than in. The vice-versa situation is true as well.
    - Remember the rate of heat energy flow (per unit area) is just the heat flux **q**. What does it mean if there is no change in heat content with time and there are no heat sinks or sources?
        - $\mathbf{q}_{\mathbf{in}} = \mathbf{q}_{\mathbf{out}}$
- Surely by know you would agree that the net flux of heat per unit volume per unit time at a point (or passing across the boundaries of an infinitesimal test cube) must be given by

$$\nabla \cdot \mathbf{q}$$

This is the left hand side of our energy rate equation. Equating both sides leads to

$$\nabla \cdot \mathbf{q} = -\rho C_p \frac{\partial T}{\partial t}$$

where the minus sign guarantees that a decrease in temperature with time will lead to a positive outflow of heat.

- But wait, there's more! We can substitute Fourier's Law of heat conduction and obtain

$$\nabla \cdot (-K\nabla T) + \rho C_p \frac{\partial T}{\partial t} = 0$$

For $K$ constant this becomes

$$\nabla^2 T - \frac{1}{\kappa}\frac{\partial T}{\partial t} = 0$$

Good grief, the LaPlacian operator has reappeared, along with a time-dependent term. This type of equation is known as a *diffusion equation*. It is everywhere in Nature and does much more than describe the transient behavior of temperature in a conductive medium. We define the *thermal diffusivity*, κ, as

$$\kappa = \frac{K}{\rho C_p}$$

(named by Lord Kelvin; Clerk Maxwell called this quantity the "thermometric conductivity", a name that didn't stick) A handy value for most rocks is $\kappa = 10^{-6}\ m^2\ s^{-1}$. One dimensionally (say, in the $z$ direction), the diffusion equation is written as

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\kappa}\frac{\partial T}{\partial t} = 0$$

- This is the one-dimensional form of the *heat equation* or the *thermal diffusion equation*.
- As you can imagine, this is the major mode of heat transfer in the crust.
- We have talked about the possibility of radioactive heat sources in the earth. Let's say the produce thermal energy at a rate of $Q$ (Watts/m$^3$). By themselves, they would produce an energy balance

$$\nabla \cdot \mathbf{q} = Q$$

We actually introduced this equation earlier in the term as an example of the divergence of a vector. By the divergence theorem:

$$\int_{\text{volume}} \nabla \cdot \mathbf{q} dv = \int_{\text{area}} \mathbf{q} \cdot \mathbf{n} da = \int_{\text{volume}} Q dv$$

in steady state all of the heat generated within a volume must pass outwards across the boundaries of that volume. If the volume is also heating or cooling, then

$$\nabla \cdot \mathbf{q} = -\rho C_p \frac{\partial T}{\partial t} + Q$$

and substituting Fourier's Law, our one-dimensional diffusion equation becomes

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\kappa} \frac{\partial T}{\partial t} = -\frac{Q}{K}$$

- This is nothing more than a statement of conservation of energy. Can you see it in the equation?
- What is the major mode of heat transfer in the mantle?
- **Convection**
    - In *convection* heat is transported mostly by the movement of material, rather than by conduction. A good example of a place where convection occurs is the atmosphere, whether it is just in a room or in the lower part of the Earth's atmosphere. When we speak of *free convection*, as occurs in the Earth's mantle, we are talking about the transport of heat by motion of the mantle. In turn, the distribution of temperature leads to density instabilities (via the coefficient of thermal expansion), which drive the mantle flow. The simpler process of *advection* is heat transport by the movement of material without considering the feedback process of the redistribution of temperature affecting the velocity of the material. "Wind chill" is a good example of the advection process.

Consider a plane *dx*-*dy* perpendicular to the *z*-axis.

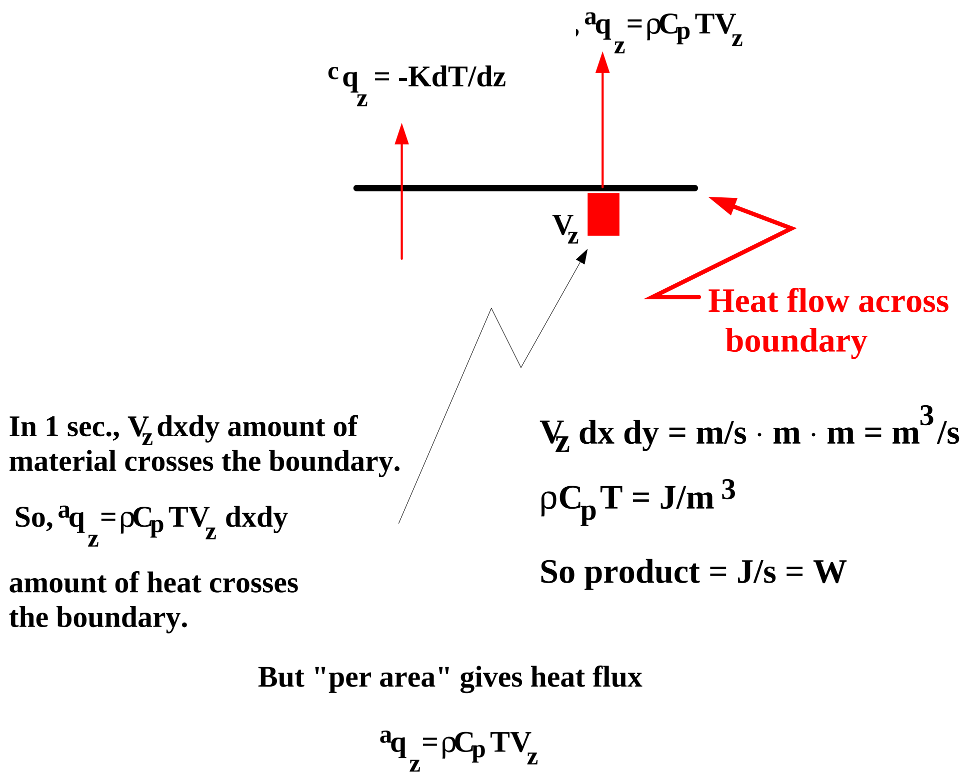


Figure 4

Figure 4 considers two ways to transport heat. In the *z* direction then

$$q_z = {}^{c}q_z + {}^{a}q_z = -K\frac{dT}{dz} + \rho C_p T V_z$$

where we note the contributions of both conductive and advective heat flux. We can use the total heat flux in our energy balance equation; the one-dimensional version would be

$$\frac{\partial q_z}{\partial z} + \rho C_p \frac{\partial T}{\partial t} = Q$$
$$\frac{\partial\left({}^c q_z + {}^a q_z\right)}{\partial z} + \rho C_p \frac{\partial T}{\partial t} = Q$$
$$-K\frac{\partial^2 T}{\partial z^2} + \rho C_p \frac{\partial\left(T V_z\right)}{\partial z} + \rho C_p \frac{\partial T}{\partial t} = Q$$
$$-K\frac{\partial^2 T}{\partial z^2} + \rho C_p \left[T\frac{\partial V_z}{\partial z} + V_z \frac{\partial T}{\partial z}\right] + \rho C_p \frac{\partial T}{\partial t} = Q$$

But if we consider that the material is incompressible, then

$$\nabla \cdot \mathbf{V} = 0$$

Only if the material is compressible will the divergence of the velocity field not vanish. One-dimensionally, the above equation is:

$$\frac{\partial V_z}{\partial z} = 0$$

We also divide out by *K* to get

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\kappa}\left[V_z \frac{\partial T}{\partial z} + \frac{\partial T}{\partial t}\right] = -\frac{Q}{K}$$

In three dimensions (and independent of coordinate system) this is

$$\nabla^2 T - \frac{1}{\kappa}\left[\mathbf{V} \cdot \nabla T + \frac{\partial T}{\partial t}\right] = -\frac{Q}{K}$$

In either equation, in order the four terms represent the flux of heat conductively across the boundaries of an infinitesimal test volume, the flux of heat by advection across the boundaries of an infinitesimal test volume, the heating or cooling of the volume, and the heat generation within the volume by heat sources. One-dimensionally, this equation is sometimes written as

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\kappa}\frac{DT}{Dt} = -\frac{Q}{K}$$

where the new operator

$$\frac{DT}{Dt} = V_z \frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

is known as the *convective derivative*. In a fixed reference frame (an Eulerian description) the time rate of change of a field quantity within a moving medium is equal to the sum of actual changes within a reference frame moving with the medium **plus** time variations due to the reference frame moving past the observer. Consider a sinusoidal magnetic field frozen into the solar wind moving past the Moon. A magnetometer on the Moon recording this signal will record a sine wave despite the fact that the field is not propagating within the solar wind. This is due to the first term of the convective derivative.

In the Earth's mantle, the advective transport of heat (due to the velocity term) is the dominant way that heat is transported.

**What is necessary (but not necessarily sufficient) for convection to occur?**

- Answer: The material must act like a **FLUID**.
- Viscosity is the important parameter in deciding whether a material acts like a solid or a fluid and on what time scale. On short time scale the mantle acts like an elastic solid (How do we know this?). On long time scales the mantle behaves as a fluid (How do we know this?). Have you ever observed any material that has this behavior?
- On what time scale do you think convection takes place through the air in a room?
- And in the mantle?