Gravity Fields - Part I

(see Chapter 2 of Lowrie)

But first, more on vector calculus

More on the gradient: Consider a surface of constant potential, $\varphi(x, y, z) = c$. Since a change in φ in a direction lying in the plane of φ must be zero, the direction of maximum change must be perpendicular to the surface (since we can decompose any arbitrary direction into two components: One lying in the plane and one perpendicular to the plane at a point). But this is the direction of $\nabla\varphi$. Therefore it follows for any potential field

$$F = \nabla\varphi$$

that the field **F** must everywhere be perpendicular to *equipotential surfaces*. This will turn out to be extremely important.

Divergence of a Vector

We introduce the *divergence of a vector*:

$$\nabla \cdot \mathbf{q} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}\right)$$
$$= \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

Note that we have the "hungry" *del* (or *nabla*) operator, which is a vector, forming a *dot product* with the vector **q**. The result is a *scalar*.

It is easiest to picture the concept of the divergence of a vector if we let **q** be fluid flow. Here we will imagine a fluid flowing through a test volume, and **q** will be a volume flow rate ($m^3\ s^{-1}$) per unit area (m^2) so that **q** has the dimensions of velocity ($m\ s^{-1}$). The divergence of a vector field tells us how much of a field is created or destroyed at any given point. So $\nabla \cdot \mathbf{q}$ tells us about how much a flow rate might change at any given point.

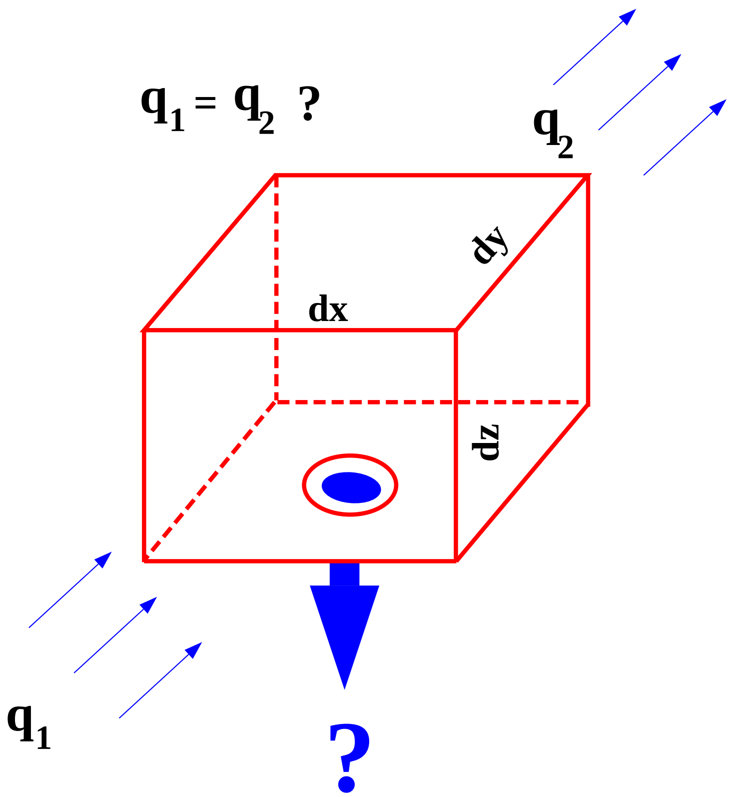

Figure 1

Figure 1 shows an infinitesimal cubical volume. The question is: Does the fluid flow rate change as it passes through the volume? If there is a "sink" within the volume, then it certainly will. For the infinitesimal cube, $\nabla \cdot \mathbf{q}$ tells us exactly how much the flow rate $\mathbf{q}$ increases or decreases as it passes through the volume. If there are no sources or sinks within the volume, then $\nabla \cdot \mathbf{q} = 0$.

It should be obvious from Figure 1 that the change in flow rate within the volume must have something to do with the amount of material passing across the surface bounding the volume. This is expressed formally as the *divergence theorem*:

$$\int_{\text{surface}} \mathbf{q} \cdot \mathbf{n} ds = \int_{\text{volume}} \nabla \cdot \mathbf{q} dV$$

where **n** is a local unit normal to the surface. **Why q·n?** This theorem states that *flux* of a vector field across a closed surface is equal to the divergence of the vector field throughout the enclosed volume. **Can you draw a picture of this?**

Here is an application of the divergence theorem. If **q** is heat flux (W m^{-2}) and at any point there is a volumetric source of heat Q (W m^{-3}) then

$$\nabla \cdot \mathbf{q} = Q$$

Then over a volume

$$\int_{\text{volume}} \nabla \cdot \mathbf{q} dV = \int_{\text{volume}} Q dV$$

but by the divergence theorem:

$$\int_{\text{surface}} \mathbf{q} \cdot \mathbf{n} ds = \int_{\text{volume}} Q dV$$

Assume the Earth, with radius R_E, can be represented by a spherical volume of constant heat sources. Then

$$q \cdot 4\pi R_E^2 = Q \cdot \tfrac{4}{3}\pi R_E^3$$

or

$$q = Q\frac{R_E}{3}$$

an answer we had obtained previously. If we measure q at the surface of the Earth, then we can use this equation to obtain a first-order approximation of the heat source concentration in the Earth.

Back to Gravity

LaPlace's Equation

Let's now consider the divergence of the gravity field. Consider a point mass M; we know that the gravitational attraction is everywhere radial and given by:

$$\mathbf{g} = \frac{GM}{r^2}\mathbf{u_r}$$

Imagine an arbitrary spherical surface of radius r surrounding the sphere. Then

$$\int_{\text{surface}} \mathbf{g} \cdot \mathbf{n} ds = \int_{\text{surface}} \frac{GM}{r^2} ds = 4\pi r^2 \frac{GM}{r^2} = 4\pi GM$$

but by the divergence theorem

$$4\pi GM = \int_{\text{surface}} \mathbf{g} \cdot \mathbf{n} ds = \int_{\text{volume}} \nabla \cdot \mathbf{g} dV = 4\pi GM = 4\pi G \int_{\text{volume}} \rho dV$$

where   is density. Equating volume integrands:

$$\nabla \cdot g(x, y, z) = 4\pi G \rho(x, y, z)$$

which says that at every point in space where there is density, there is a non-zero divergence of the gravity field. We can now substitute in terms of the gravitational potential:

$$\mathbf{g} = -\nabla \Phi$$

Notice that we are now paying attention to signs. The gravity vector points in the opposite direction of increasing gravitational potential. Substituting,

$$\nabla \cdot \nabla \Phi(x, y, z) = -4\pi G \rho(x, y, z)$$

If we write out the $\nabla \cdot \nabla$ operator in Cartesian coordinates, it is

$$\nabla \cdot \nabla = \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} \right) \equiv \nabla^2$$

so

$$\nabla^2 \Phi(x, y, z) = -4\pi G \rho(x, y, z)$$

which is known as *Poisson's Equation*. Outside regions of density (at the surface of the Earth or above it),

$$\nabla^2 \Phi(x, y, z) = 0$$

This is ***LaPlace's Equation*** for gravitational potential. LaPlace's equation shows up again and again and again in Earth Forces for different potential functions: hydraulic head in aquifers, steady state heat flow, velocity of inviscid fluids, the magnetic field, etc. LaPlace's equation will become your friend.

The Earth's Gravity Field

We have already talked about the gravitational attraction of the Earth as if it were a perfect sphere. It is about 9.8 m s^{-2} = 980 cm s^{-2} = 980 gals = 980,000 mgal. This is the average gravitational attraction of the Earth. The largest departure from sphericity is due to the oblateness of the Earth resulting from rotation. Gravitational attraction is larger at the poles than at the Equator. **Why is that?** This effect is about 0.001 times the mean (spherical attraction) of the Earth. Gravity anomalies are superposed on this that are typically 0.0000001 to 0.0001 times the mean attraction of the Earth (1 - 100 mgal). These can be measured with very sensitive instruments called gravity meters (down to 0.01 mgal).

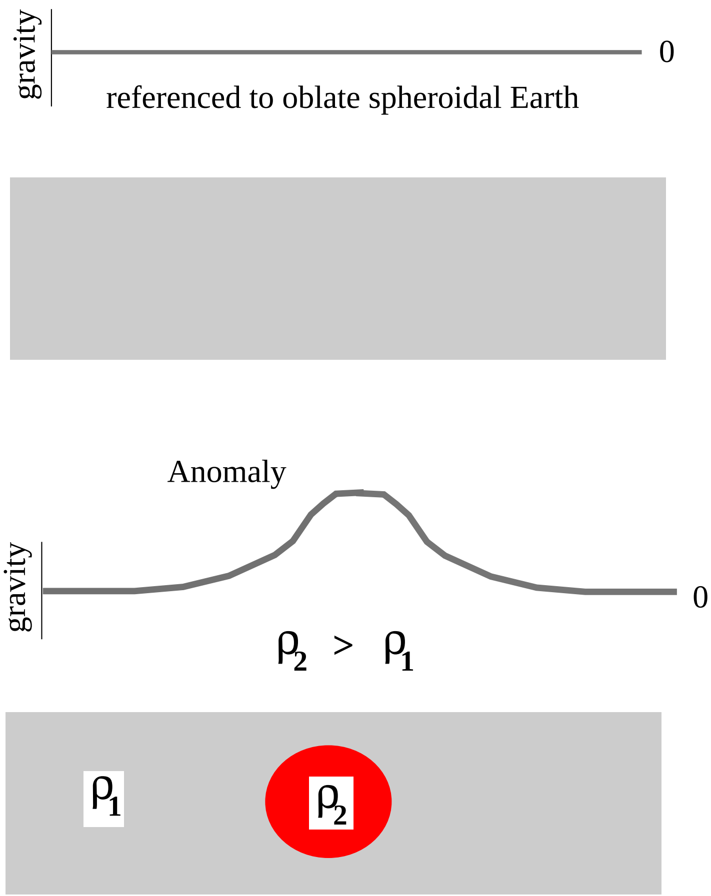

Figure 2

The gravitational potential field of the Earth can be expressed as a solution to LaPlace's equation. The solution is an infinite series that looks like

$$\Phi(r,\theta,\varphi) = \frac{GM}{r} \sum_{n=1}^{\infty} \left[\begin{array}{c} \text{messy terms depending on lat., long.,} \\ \text{\& indexed to } n \end{array} \right]$$

Instead of x, y, and z, we use a spherical notation or radius (r), latitude (θ), and longitude (φ). The leading term is 1, representing the very dominant contribution of a spherical Earth. This is known as the "central mass" term. All additional terms are much smaller than 1. The second term is on the order of 0.001 and is the second most important contribution. To a very good approximation, the Earth behaves hyrdostatically. Because of the rapid rotation of the Earth, density interfaces are of the form of oblate spheroids. The second term is essentially the effect of oblateness. All the higher order terms represent anomalies, such as the type we described in Figure 2.

Geoid

The total potential, when you are attached to a planet is due to the sum of the internal density distribution and centrifugal force. The *geoid* is a surface of constant potential. There are an infinite number of these, but by convention, mean sea level is chosen for Earth. The dominant terms of the geoid, as referenced to the central mass term, are the effects of oblateness and centrifugal force. We can call this the *reference geoid*. But all other effects due to the mass distribution of the Earth show up as anomalies on the reference geoid.

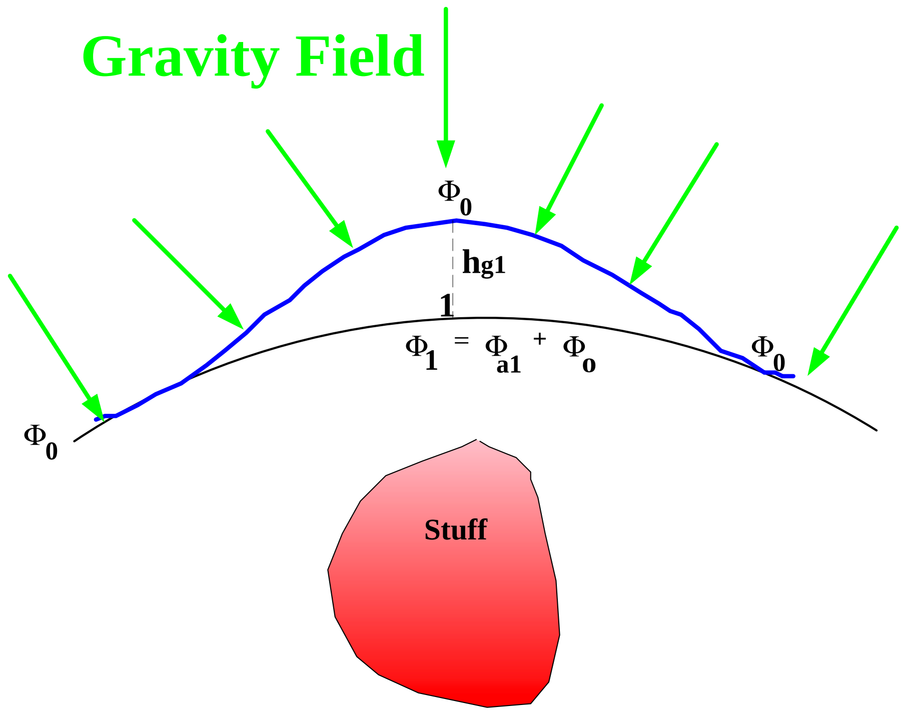

Figure 3

We can define the height of the geoid in the following way. Define Φ_0 as the total potential. Far away from some anomalous mass ("stuff") it corresponds to the reference geoid, which is shown as the black curve. Over an anomalous mass a surface of constant potential must mover farther away from the mass. At the reference level the potential must be increased by Φ_{a1}. So we have

$$\Phi_0 = \Phi_1 - \frac{\partial \Phi_1}{\partial r} h_{g1} = \left(\Phi_{a1} + \Phi_0\right) - \frac{\partial \Phi_1}{\partial r} h_{g1}$$

This equation comes about from a linear extrapolation (the derivative is the slope). If you prefer, you may also view this as a two term *Taylor Series expansion*. Solving for the anomalous potential

$$\Phi_{a1} = \frac{\partial \Phi_1}{\partial r} h_{g1} \approx \frac{\partial \Phi_0}{\partial r} h_{g1} \approx g_0 h_{g1}$$

We recognize the derivatives as the radial component of the gravity field. The approximation comes about because the mean gravity field is much, much larger than the anomaly. So for the *geoid height*:

$$h_{g1} = \frac{\Phi_{a1}}{g_0}$$

We can measure the geoid height over the oceans with a very precise altimeter on a spacecraft.

(See the oceanic geoid map in PowerPoint file)

Reduction of Gravity Data

The gravity anomalies we are trying to map in land surveys are buried in a much larger signal (g_0 ≈980 cm/s^2 = 9.8x10^5 mgal). The trick is to pull it out. Even second order global effects will swamp local anomalies. Following are the types of data reduction that must be done with data acquired with a gravity meter.

Latitude Effect. The gravity field associated with the reference geoid (central mass term plus oblateness) can be expressed as a function of latitude θ by

$$g(\theta) = g_e\left(1 + a\sin^2\theta\right)$$

where g_e is the gravity at the equator. For Earth, the constant a is positive, so gravity is greater at the poles than at the equator. **Why is this?**

The equation shows that when you conduct a gravity traverse, there will be an effect just due to changing latitude. If we differentiate this equation with respect to latitude, we can obtain the *latitude correction*, which is about 0.8 mgal/km (in the N-S direction). Depending on the areal extent of a survey, this effect can be important.

Free-Air Correction. All additional corrections are designed to "reduce the measurement to the geoid." **What does that mean?** As you move away from the geoid the gravity field decreases. It obviously depends on what latitude you're at. The correction is traditionally done only with the central mass term, however:

$$g_0 = \frac{GM}{r^2}$$

The rate of decrease of the field is

$$\frac{dg_0}{dr} = -\frac{2GM}{r^3} = -\frac{2g_0}{r}$$

When the radius used is the equatorial radius, then the gravity field decreases as you move above the geoid by 0.3085 mgal/m. Therefore , you use the elevation of the "gravity station" to add this correction to the data, as long as you are above sea level. The resulting gravity is known as "*free-air gravity*".

Bouguer correction. The Bouguer correction accounts for attraction of material between the point of observation and the geoid. If we assume that an observation is made on a plateau of height *h* above the geoid and that the plateau extends to infinity in all horizontal directions, then the attraction of this crustal "slab" is

$$g_{\text{slab}} = 2\pi\rho_c G h$$

where ρ_c is the density of the slab. Using the "canonical" value of ρ_c = 2670 kg/m^3, this formula shows that gravity increases with elevation by 0.112 mgal/m. This correction is always subtracted. **Why?** This is sometimes called the "*simple Bouguer correction*". The resulting gravity is known as "*Bouguer gravity*".

Terrain Correction. The slab formula is nothing more than a first order attempt to account for the gravitational attraction of topography. The gravity field measured is the sum of topography plus what is inside the Earth. We already know what the topography is, so we need to remove its effect in order to isolate the interior density anomalies. The terrain correction accounts for departures from a perfect slab. The result is always added. **Why?** This correction is sometimes called the "complete Bouguer correction" and the result is called "complete Bouguer gravity".

Drift correction. Gravity meters are not perfect; they tend to *drift* in their readings. Most of this is caused by temperature effects – specifically the spring constant changes with temperature. Additional drift is caused by tidal effects. The better gravity meters have good "temperature compensation." We usually have to occupy a base station a number of times during a survey and establish a *drift curve* with which to correct the data. How often the base station needs to be reoccupied depends on the rate of drift, of course.

Modeling

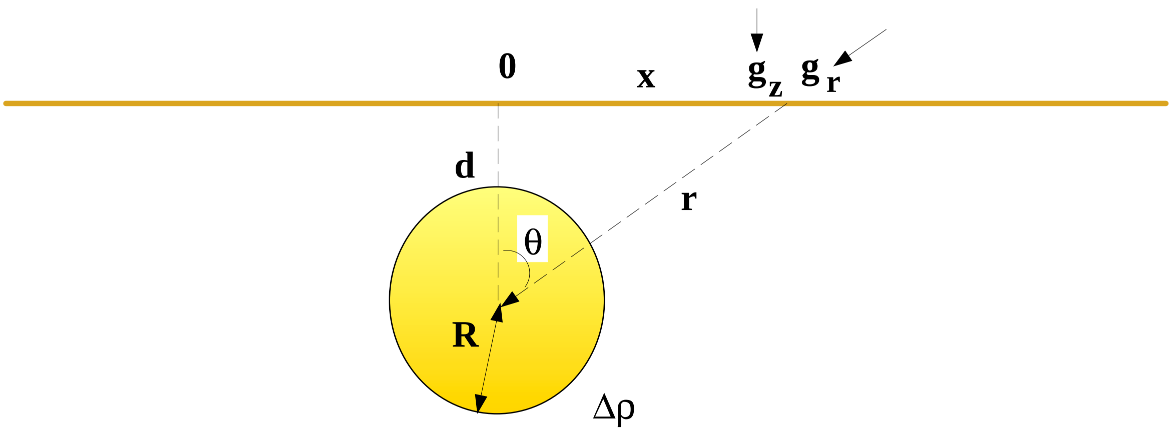

Figure 4

Most activities in geophysics will involve gathering some data and fitting a model to the data. The model is a simplified version of reality. Usually we try to estimate the parameters of the model that cause it to have the best fit to the data. You recall that the gravity anomaly of a sphere is GM/r^2, in a radial direction from the sphere. Most gravity meters measure vertical gravity. Consider a sphere at a depth d below the surface. The sphere has radius R, and a density contrast $\Delta\rho$ with its surroundings. As a function of traverse position x, the vertical gravity anomaly of a sphere is

$$g_z = \tfrac{4}{3}\pi R^3 G\Delta\rho \frac{\cos\theta}{\left(x^2+d^2\right)} = \tfrac{4}{3}\pi R^3 G\Delta\rho \frac{1}{\left(x^2+d^2\right)} \frac{d}{\left(x^2+d^2\right)^{1/2}}$$

or

$$g_z = \tfrac{4}{3}\pi R^3 G\Delta\rho \frac{d}{\left(x^2+d^2\right)^{3/2}}$$

What size anomaly might you expect? Let's plug in some numbers. Let's say you are looking for a cave or a big limestone cavity. You an take $\Delta\rho$ = -2.5 gm/cm^3 = -2500 kg/m*3*. Take R = 100 m and d = 120 m. Plugging into the above formula gives -5 mgal, easily measurable.

The parameters of this model are R, $\Delta\rho$, and d. You can fit this model to field data by adjusting these 3 parameters to get the best match. Note that you will never be able to solve for R and $\Delta\rho$ independently. There is a trick we can use to solve for d. The anomaly will fall off from a peak directly over the sphere. At some distance x, which we call $x_{1/2}$, the anomaly has fallen off to one-half its peak value. Equating the anomaly at $x_{1/2}$ to have of the peak value gives:

$$\frac{d}{\left(x_{1/2}^2 + d^2\right)^{3/2}} = \frac{1}{2}\frac{1}{d^2}$$

or

$$d = 1.3x_{1/2}$$

The sphere is the simplest possible shape that we can imagine. For more complicated and realistic shapes, we have to solve LaPlace's equation for the geometry we desire.

Here is one example. A particular oceanic geoid map has not named the units. Is the geoid in cm or m? A trench geoid anomaly is either about -60m or -60cm. How could you figure it out? Let's approximate the trench with a cylinder. We can solve LaPlace's equation for an infinitely long circular cylinder with the result that

$$\Phi = 2GM \ln R$$

where M is the mass per unit length of the cylinder, located at (x_0, z_0), and

$$R = \sqrt{\left(x - x_0\right)^2 + \left(z - z_0\right)^2}$$

If the cylinder has radius a, then the geoid height is given by

$$h_g = \frac{2\pi a^2 \Delta\rho G}{g_0} \ln R$$

The trench is about 75 km wide and 4 km deep. The mean radius is one-half the square root of 75x4 = 8.7 km. R is the depth of the trench, or 4 km. Using g_0 = 9.8 m s^{-2} and $\Delta\rho$ = -2500 kg m^{-3}, then

$$h_g = \frac{-2\pi \cdot (8700)^2 \cdot 2500 \cdot 6.673\text{x}10^{-11}}{9.8} 8.3 = -67 \text{ m}$$

So we conclude that the units are meters.