

Viscous Flow in Earth Forces — Part III

Plume behavior

We can examine solutions to the Navier-Stokes equations to study plume behavior in the mantle. Plumes are thought to arise from a hot region at the base of the mantle. Since this region is hotter than the mantle region above it, this region is also less dense than the mantle above it. This region is thus inherently unstable. We assume that an instability breaks out and forms in a spherical shape (the “plume head”). It will eventually start to rise off the hot layer. The vertical velocity of ascent is given by a solution to the Navier-Stokes equations at low Reynolds Number (acceleration = 0). We assume that the viscosity of the rising plume is much less than the viscosity, η_m , of the surrounding mantle. In this case

$$V_z = \frac{1}{3} \frac{a^2 g_0}{\eta_m} \Delta\rho \quad (1)$$

where a is the radius of the sphere and $\Delta\rho$ is the density contrast between the plume head and the surrounding mantle. Note that the larger the plume head (depends on a^2) or the bigger the density contrast, the larger is the velocity. Note also that high viscosity of the surrounding mantle will impede flow.

Let’s tie this spherical plume head to the overall plume hypothesis: Plumes arise from instabilities in the hot thermal boundary layer at the base of the mantle. As you know, the hot thermal boundary layer exists because heat from the core must be transferred into the mantle by conduction. The initial instability breaks out as a quasi-spherical plume head of hot material that rises buoyantly through the mantle. Trailing behind is a plume tail (Figure 1).

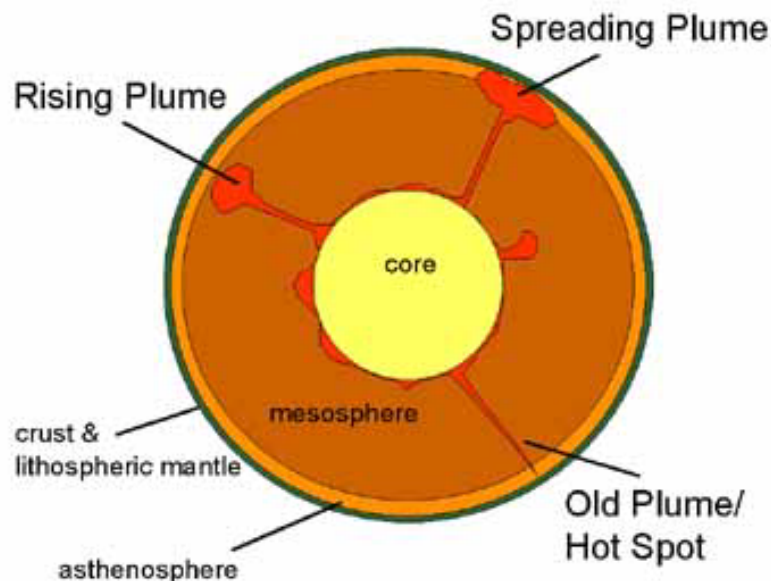


Figure 1.

As the plume starts to form at the hot boundary, it will not begin to ascend as long as its velocity, V_z , is less than its growth rate da/dt . As the upward velocity increases with a^2 , eventually it will start to rise. The plume is fed at a constant rate Q in m^3/s . Clearly

$$4\pi a^2 \frac{da}{dt} = Q \quad (2)$$

and the plume will start to rise when $V_z = da/dt$. The radius at this time [equating equations (1) and (2)] is

$$a_0 = (3Q\eta_m / 4\pi g_0 \Delta\rho)^{1/4} \quad (3)$$

and this takes a time obtained from the volume at a_0 $[(4/3)\pi a_0^3]$ divided by the feeding rate Q :

$$t_0 = (4\pi / 3Q)^{1/4} (\eta_m / g_0 \Delta\rho)^{3/4} \quad (4)$$

As the plume starts to rise, there is a trailing cylindrical conduit (the “plume tail”). See Figure 2.

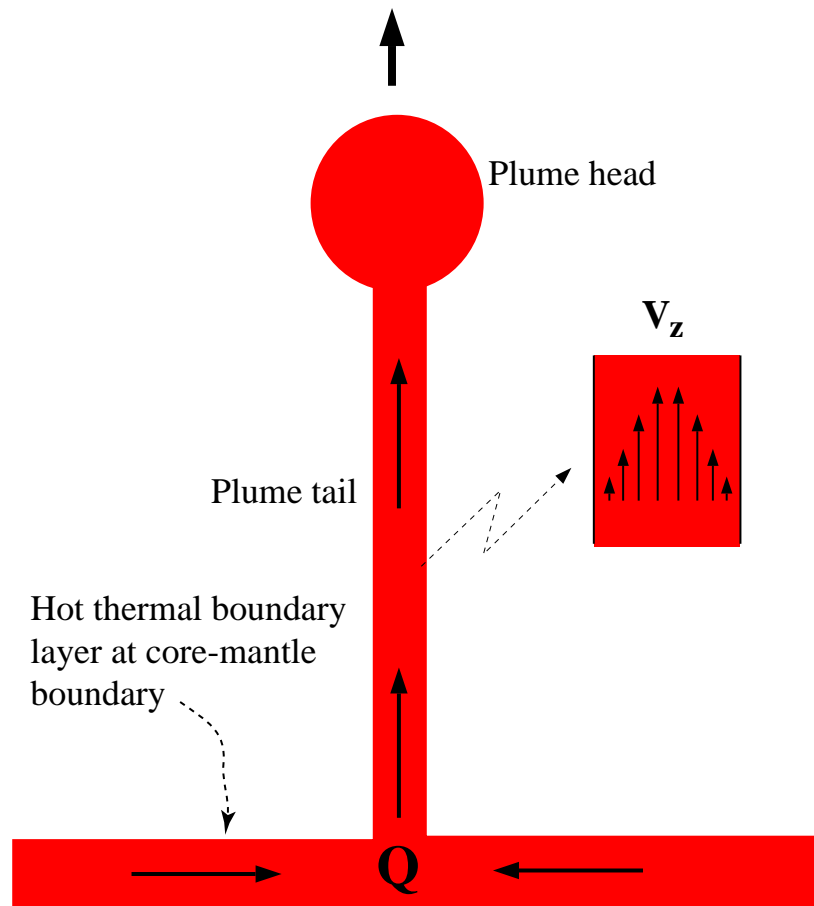


Figure 2.

We assume that flow is driven by the body force in the conduit created by the density contrast, $\Delta\rho$, of the fluid with its surroundings. Flow is described by the Navier-Stokes equation:

$$\eta_p \nabla^2 \mathbf{V} - g_0 \Delta\rho \mathbf{u}_z = 0 \quad (5)$$

where η_p is the viscosity of the hot plume material. Velocity in the conduit is vertical and varies only in the horizontal direction (See Figure 2.). The natural coordinate system to describe this problem is cylindrical. Instead of x - y - z coordinates, we have r - θ - z coordinates, where r is radial to the axis of the cylinder, θ is an angular distance around the cylinder, and z is still the vertical coordinate. Since there is no z dependence, and the problem is axially symmetric, then the Navier-Stokes equation in cylindrical coordinates is (only the component V_z):

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dV_z}{dr} \right) = \frac{g_0 \Delta\rho}{\eta_p} \quad (6)$$

Keep in mind that the vertical velocity in the pipe is not the same as the sphere velocity. By integration, this has a solution

$$V_z = \frac{g_0 \Delta\rho}{4\eta_p} (-r^2 + A \ln r + B) \quad (7)$$

To avoid a singularity at $r = 0$, $A = 0$. At the conduit boundary, $r = r_0$, and the velocity in the tail goes to 0 there. This allows determination of the constant B . The solution is then:

$$V_z = \frac{g_0 \Delta\rho}{4\eta_p} (r_0^2 - r^2) \quad (8)$$

Integrating this equation gives the volume flux through the conduit. Note that this is a velocity time a cross sectional area, or m/s times $\text{m}^2 = \text{m}^3/\text{s} = \text{volume flux}$. We have

$$Q = \frac{g_0 \Delta\rho}{4\eta_p} \int_0^{r_0} (r_0^2 - r^2) 2\pi r dr \quad (9)$$

from which one obtains the radius of the conduit:

$$r_0 = (8\eta_p Q / \pi g_0 \Delta\rho)^{1/4} \quad (10)$$

Now as the plume head rises with velocity V_z , the plume tail must lengthen to keep up with it. The rate of volume change in the pipe is just its cross sectional area times its rate of lengthening, which is just V_z . This is then, from the buoyant sphere formula [equation (1)]

$$\pi r_0^2 V_z = \pi r_0^2 \frac{a^2 g_0 \Delta\rho}{3\eta_m} \quad (11)$$

Therefore the plume grows as

$$4\pi a^2 \frac{da}{dt} = Q - \pi r_0^2 \frac{a^2 g_0 \Delta\rho}{3\eta_m} \quad (12)$$

What does this equation say? It says that the plume head grows at a rate proportional to the volume flux, Q . But as the plume rises, the plume tail must take up some of the fluid, which detracts from the growth of the plume head.

Figure 4 shows some solutions to equation (12).

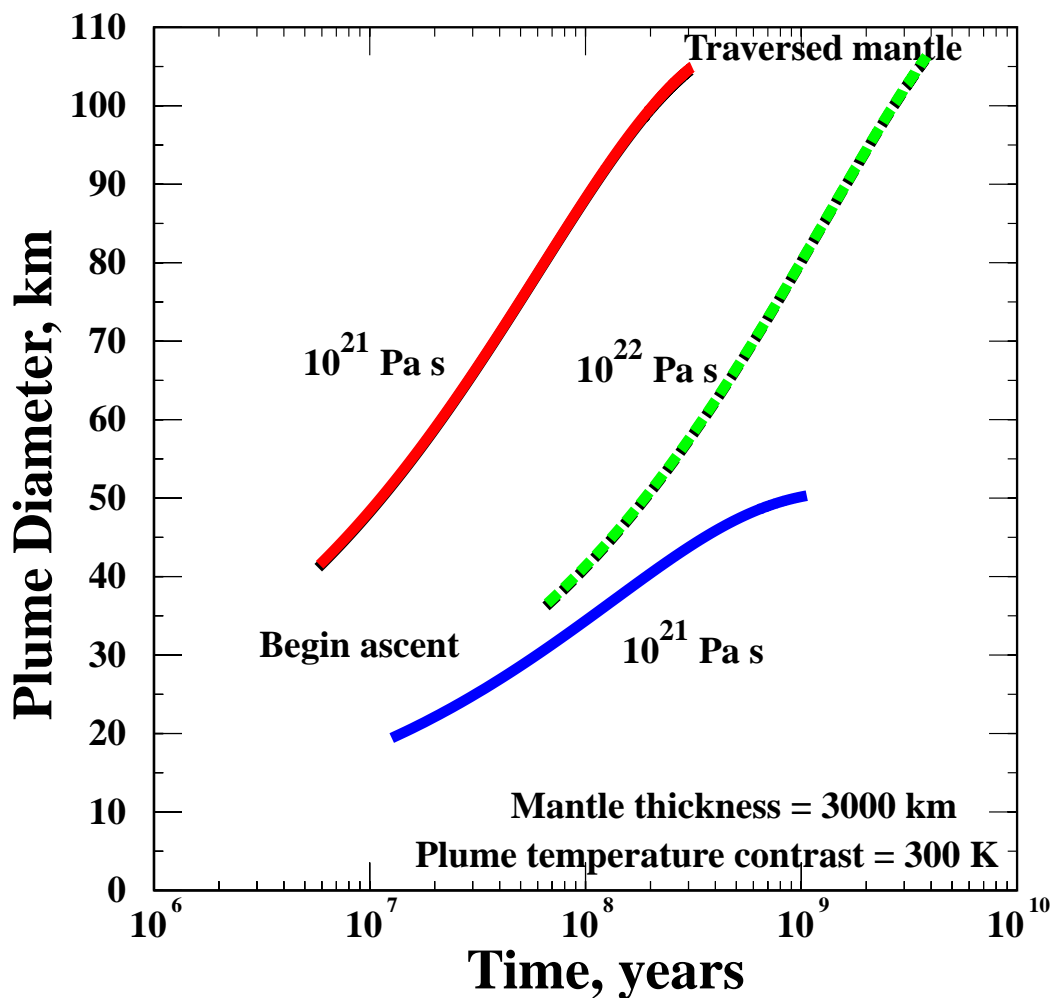


Figure 4. Blue curve has a lower value of Q than for the other two curves, for which Q is the same.

What can we learn here about what types of plumes might traverse the entire mantle to create hotspots?