Viscous Flow in Earth Forces — Part II
See Chapter 6 of Lowrie

On to the Navier-Stokes Equation

The Navier-Stokes equation is the most famous equation in fluid dynamics. It is the differential equation for the velocity of a viscous fluid. In principle, it is easy to see where it comes from. Start with the equilibrium equations:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} + X = 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0 \end{aligned} \tag{1}$$

Then express stress in terms of velocity:

$$\begin{aligned} \sigma_{zx} &= \sigma_{xz} = \eta\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right) \\ \sigma_{xx} &= -P + 2\eta\frac{\partial V_x}{\partial x} \\ \sigma_{zz} &= -P + 2\eta\frac{\partial V_z}{\partial z} \end{aligned} \tag{2}$$

We have thrown in a new term here, P, which is simple fluid pressure. Even in an inviscid fluid ($\eta = 0$), a fluid is subject to a hydrostatic pressure. In general, P is just the average of all the normal stresses:

$$P = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \tag{3}$$

In a fluid at rest these three stresses are the same.

If we now substitute equation (2) into (1),we get

$$\begin{aligned} \frac{\partial}{\partial x}\left[-P + 2\eta\frac{\partial V_x}{\partial x}\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)\right] + X = 0 \\ \frac{\partial}{\partial x}\left[\eta\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[-P + 2\eta\frac{\partial V_z}{\partial z}\right] + Z = 0 \end{aligned} \tag{4}$$

This can be rearranged, if the viscosity, η , is constant into

$$\eta\left[\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial z^2}\right]+\eta\frac{\partial \xi}{\partial x}-\frac{\partial P}{\partial x}+X=0$$
$$\eta\left[\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial z^2}\right]+\eta\frac{\partial \xi}{\partial z}-\frac{\partial P}{\partial z}+Z=0 \tag{5}$$

These are the two-dimensional Navier-Stokes Equations for constant viscosity. The first equation is the *x* component and the second equation is the *z* component. The term ξ is given by

$$\xi=\frac{\partial V_x}{\partial x}+\frac{\partial V_z}{\partial z}=\nabla\cdot\mathbf{V} \tag{6}$$

The only way that a fluid can have non-zero divergence $\nabla\cdot\mathbf{V}$ in the direction it is flowing is if it is compressible. In most geological and geophysical applications, we assume that compressibility can be ignored ($\nabla\cdot\mathbf{V} = 0$), so the terms containing ξ drop out of the Navier-Stokes equations.

Equation (5) can be expressed in a compact vector form:

$$\eta\nabla^2\mathbf{V}-\nabla P+\mathbf{f}=0 \tag{7}$$

where **f** is the body force vector. This is a big-time equation that is used everywhere to describe fluid flow (ocean, atmosphere, core, mantle). This is still nothing more than **F** = 0 with a stress-velocity relationship. Sometimes we have to include the acceleration term m**a**. See below. As an example of the Navier-Stokes equation, consider convection in the mantle. **f** is a gravitational body force $\rho g_0\mathbf{u_z}$, where $\mathbf{u_z}$ is a unit vector in the z direction. But density variations are driven by temperature variations $\rho_0\alpha T$, where ρ_0 is a reference density of the mantle, and α is the volume coefficient of thermal expansion. So we can write equation (7) as

$$\eta\nabla^2\mathbf{V}-\nabla P+g_0\rho_0\alpha T\mathbf{u_z}=0 \tag{8}$$

You may we recall that when we studied heat flow, the diffusion equation for temperature, when we included the possibility of advection of temperature by a velocity field, **V**, looked like this:

$$\nabla^2 T-\frac{1}{\kappa}\left[\mathbf{V}\cdot\nabla T+\frac{\partial T}{\partial t}\right]=-\frac{Q}{K} \tag{9}$$

where κ is diffusivity, *K* is thermal conductivity, and *Q* is the heat source concentration. Equations (8) and (9) are the coupled differential equations for the velocity field and temperature in a convecting mantle. We have to solve these equations simultaneously. We have been talking about these coupled equations all semester without actually introducing them. Many make there living solving these equations.

A Little Dimensional Analysis

We can gain some insight into solutions of the Navier-Stokes equations by using a dimensional analysis. We look at the characteristic dimensions of all of the terms in an equation and find their interrelationships. We will try this out on the z-component of the Navier-Stokes equations. Let us consider a one-dimensional problem of the behavior of the vertical component of velocity in response to gravity. Remember that $\partial/\partial z$ has dimensions of $1/L$, where L is a characteristic length scale. Also remember that the gravity body force is $\rho g_0 \mathbf{u_z}$. We omit the pressure term, P, term because pressure cannot drive viscous flow. Thus we have for the z-component

$$\eta \frac{\partial^2 V_z}{\partial z^2} - \rho g_0 = 0 \tag{10}$$

Dimensionally this becomes

$$\eta \frac{V_z}{L} = \rho g_0 \tag{11}$$

or

$$V_z = \frac{\rho g_0 L^2}{\eta} \equiv \frac{L}{\tau_d} \tag{12}$$

where dimensionally, velocity is expressed as the characteristic length scale, L, divided by a characteristic time, τ_d. Rearranging

$$\tau_d = \frac{\eta}{\rho g_0 L} \tag{13}$$

Note that the numerator has dimensions of Pa s, whereas the denominator has units of Pa. This result tells us that the characteristic response of a viscous fluid to a gravity force increases with viscosity, but decreases with the length scale. We can compare this "back-of-the-envelope" result with the formal solution to the glacial rebound problem. See below.

What ever happened to acceleration?

We have blithely dropped the acceleration term. Let's reintroduce it to the Navier-Stokes equation to see how important it is. We have

$$\rho \mathbf{a} = \rho \frac{\partial \mathbf{V}}{\partial t} = \eta \nabla^2 \mathbf{V} - \nabla P + \mathbf{f} \tag{13}$$

Note that these equations are really in force per unit volume, so we use density instead of mass.

Dimensionally, the acceleration term on the left-hand-side is

$$\frac{\rho V^2}{L} \tag{14}$$

and the force terms on the right-handsides are dimensionally

$$\frac{\eta V}{L^2} = \frac{\nu \rho V}{L^2} \tag{15}$$

where $\nu = \eta / \rho$ is the kinematic viscosity. The ratio of left-hand-side to the right-hand-side is

$$\mathrm{Re} = \frac{\rho V^2 / L}{\nu \rho V / L^2} = \frac{VL}{\nu} \tag{16}$$

which is called the Reynolds number. High Reynolds number behavior is characterized by turbulent flow, whereas low Reynolds number behavior is characterized by laminar flow. If we neglect completely accelerations terms, we are saying that turbulence is not important; i.e., the Reynolds number is very low. We can plug some numbers in for the mantle: V = 10 cm yr^{-1} $\approx 0.1/(\pi\, 10^7)$ m s^{-1}; ν = 10^{21} Pa s^{-1}/3000 kg m^{-3} = 10^{21}/3000 m^2 s^{-1}; L = 1000 km = 10^6 m. So,

$$\mathrm{Re} = \frac{10^{-1}}{\pi \cdot 10^7} \times 10^6 \times \frac{3\text{x}10^3}{10^{21}} \approx \frac{10^8}{10^{28}} = 10^{-20} \tag{17}$$

Yes, indeed, the acceleration term can be neglected! Note also that the Reynolds number itself is dimensionless.

The Glacial Rebound Problem

We have discussed glacial rebound before. Large ice sheets in northern Europe (Fennoscandia) and Hudson Bay melted rapidly about 10,000 years ago. Prior to that time the ice sheets were in approximate isostatic equilibrium. When the ice sheets melted, the lithosphere was out of equilibrium and began to rise to re-attain equilibrium (Figure 1)

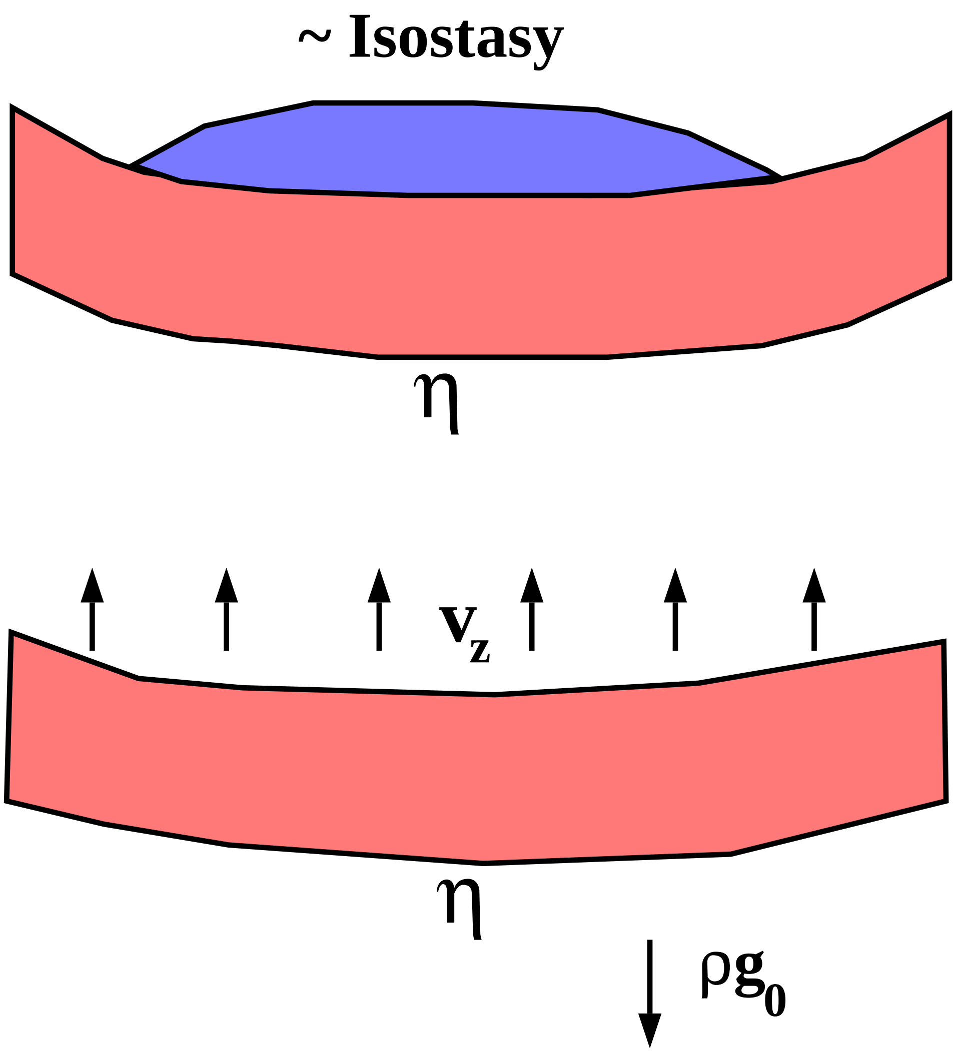

Figure 1.

Figures 2 below shows the uplift rate of the surface in Fennoscandia. The indication is that equilibrium has not been reached yet since the removal of the ice.

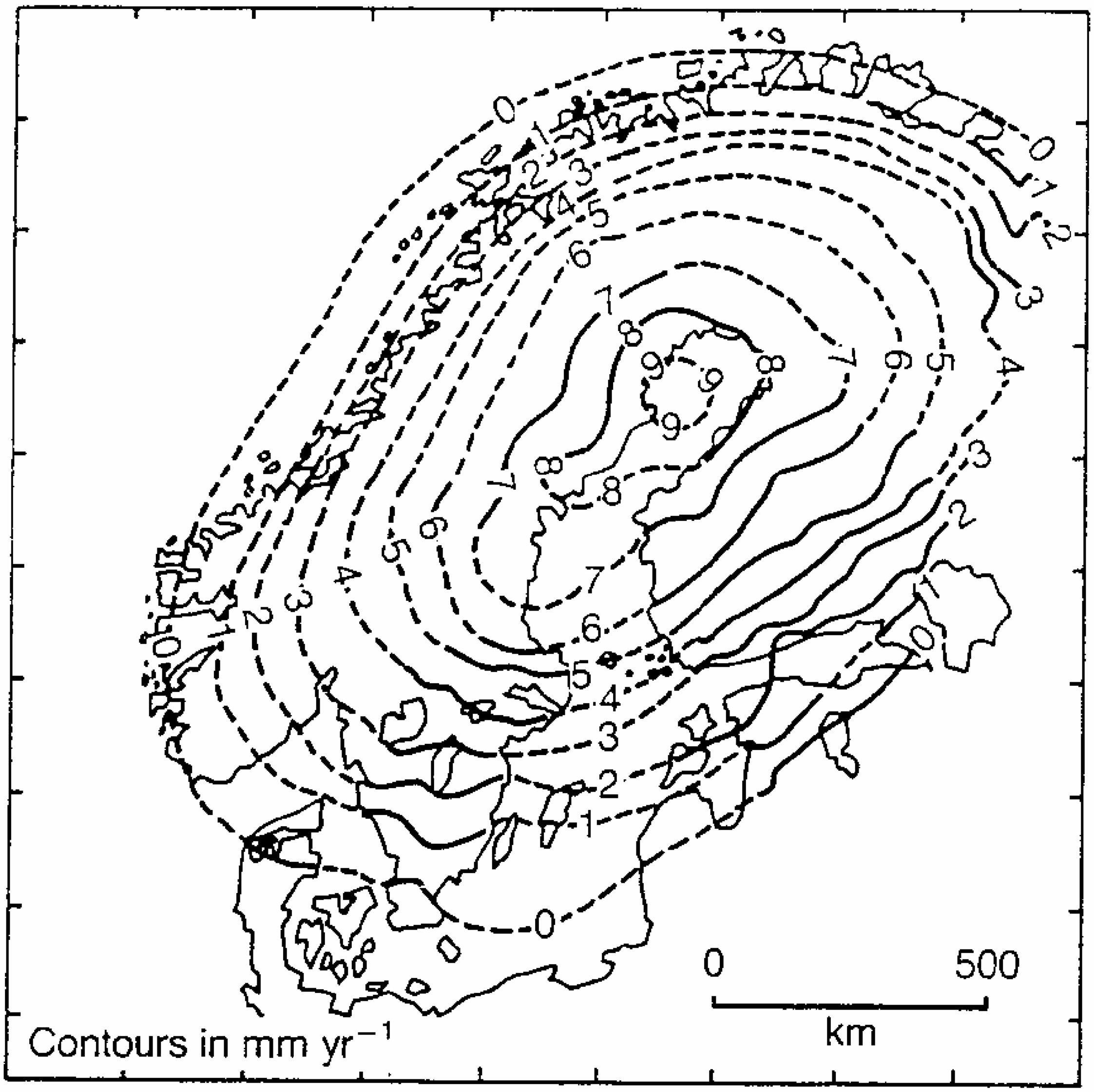

Figure 2. Uplift rate in northern Europe.

Figure 3 shows how the uplift rate is determined. Sea level in Fennoscandia is connected globally to the entire ocean system and can be taken as fixed relative to the surrounding land, which is rising.

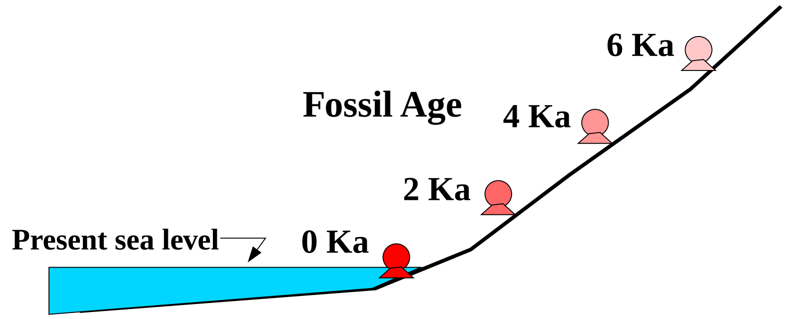

Figure 3.

In order to solve the Navier-Stokes equations easily, we have to break the problem down into its wavelength components (Figure 4). If $w(\lambda)$ is vertical deflection as a function of wavelength λ (and time t), then it can be shown that a solution to the Navier-Stokes equation is

$$V_z(\lambda) = \frac{dw(\lambda)}{dt} = -\left(\frac{\rho g_0 \lambda}{4\pi\eta}\right) w(\lambda) \tag{18}$$

Equation (18) is a simple first order differential equation in time, with solution:

$$w(\lambda, t) = w_0(\lambda)\mathrm{e}^{-t/\tau_r} \tag{19}$$

where w_0 is the initial deflection of the lithosphere, which will decay away to zero with a time constant

$$\tau_r = \frac{4\pi\eta}{\rho g_0 \lambda} \tag{20}$$

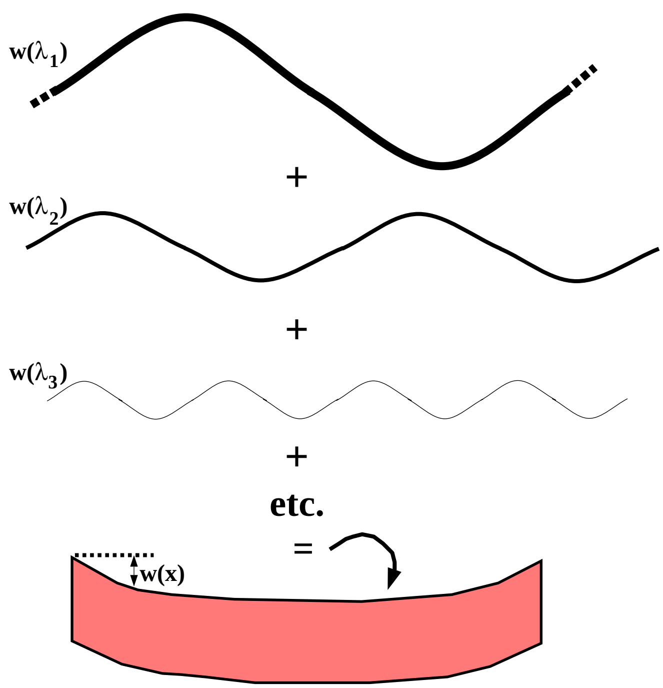

Figure 4.

The time constant τ_r is the "glacial rebound time constant" and allows a determination of the viscosity of the mantle via equations (19) and (20).

Let's compare this time constant with that which we found from dimensional analysis:

$$\tau_r = \frac{\eta}{\rho g_0 \left(\frac{\lambda}{4\pi}\right)}, \quad \text{rigorous solution}$$
$$\tau_r = \frac{\eta}{\rho g_0 L}, \quad \text{dimensional analysis} \tag{21}$$

You can see that the rigorous solution, which is pretty involved to obtain, and the solution from dimensional analysis have the same dependence on parameters. The length scale L corresponds to wavelength divided by 4π.

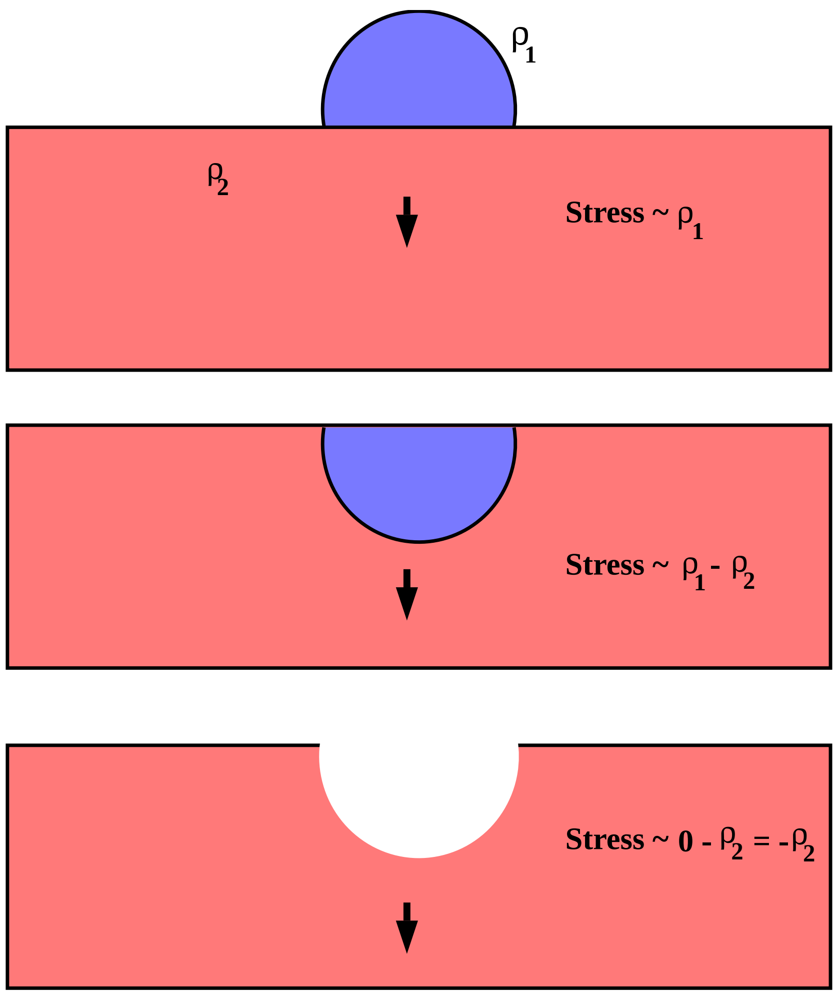

Figure 5.

In Fennoscandia and Hudson Bay, both the amount of deflection, $w(\lambda)$, and the rate of uplift, $dw(\lambda)/dt$, are measurable, and so the viscosity of the interior can be estimated. To what depth does the viscosity correspond? The deflection of the ground represents a load on the interior. To see that a hole in the ground is a load on the interior, consider a hemispherical mass on the surface (Figure 5). In the absence of this mass, there is no flow. There will only be flow if there are density contrasts whose surfaces are not level, or horizontal. We already saw this when we studied the sloping glacier and found that a sloping surface gives rise to a gravity pressure gradient. Level surfaces are defined as those where the gravitational effect is the same anywhere on the surface. As you know, they are sometimes called equipotential surfaces. Every volume bounded by a surface that does not lie on an equipotential surface experiences stresses like the one in the glacier. If the volume can flow, its will flow until it bounding surface is an equipotential surface. If the material acts like a viscous fluid, this will happen for sure, but at may take awhile, depending on viscosity and wavelength [equation (20)].

When we add the hemispherical load, the ground is deflected by the load, and stresses are set up in both the load and in the ground (Figure 5). The load is proportional to the volume of the hemisphere ($2/3\ \pi r^3$) and its density, ρ_1. If the same load is in the ground, the load is proportional to (ρ_1 - ρ_2), for if the density contrast vanishes, there is no load. Finally, we can consider a hole in the ground, where the density contrast is ($0 - \rho_2$) = $-\rho_2$. This negative load will impart stresses that are opposite in side for the case of a positive load.

It can be shown that the stresses imparted by a load are proportional to an exponential function of depth

$$\sigma \sim e^{-\frac{2\pi}{\lambda}z} \qquad (22)$$

So the stresses, which cause flow in the crust and mantle, become weaker with depth. This flow is upward to remove the hole. The rate of flow depends inversely on the viscosity

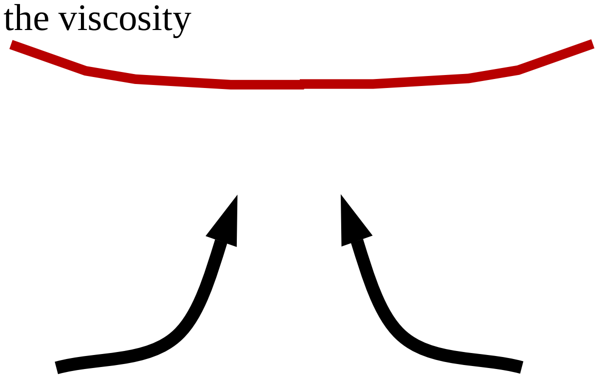

Figure 6. The load of the surface depression imparts stress at depth. The stress drives viscous flow upward to remove the hole.

But notice that the larger the wavelength, the deeper the stresses penetrate, and so the deeper must be the flow taking place in response to the stresses. This tells us that the longer wavelengths tell us about the viscosity of the middle and lower mantle, while the shorter wavelengths tell us about the viscosity of the upper mantle and lower crust. So it is possible that by doing a wavelength by wavelength analysis of the post glacial rebound, we can obtain a viscosity profile of the mantle, $\eta(z)$. This will end up telling us a lot about the way the mantle convects. To be sure, the first time the problem was done, it assumed a constant viscosity. The value obtained, about 10^{21} Pa s, is still a good number for the upper mantle. This values allowed the calculation of a Rayleigh number, which showed that the mantle must be convecting!