Viscous Flow in Earth Forces Part I
See Chapter 6 of Lowrie

Introduction

The consideration of viscous flow of the solid Earth dominates geodynamics. Consider:

1. mantle convection
2. flow in the lower crust (see folded gneisses and schists)
3. flow of ice and rock glaciers
4. landslides and mudflows
5. lava flows
6. folding of strata

Basic Relationships

We have already discussed that just as stress is related to strain through Young's modulus, stress is related to strain rate through viscosity. Here are some parallels between strain, ε_{ij}, and strain rate, $\dot{\varepsilon}_{ij} = d\varepsilon_{ij}/dt$. We examine the relationships in two-dimensions. We know that strain is given in terms of spatial gradients of displacement (*u* and *v* are the *x* and *z* components, respectively, of displacement in an *x*-*z* system):

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{zx} \\ \varepsilon_{xz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \tag{1}$$

In viscous flow, the strain rate tensor is expressed in terms of gradients of velocity (V_x and V_z are the *x* and *z* components of velocity, respectively):

$$\dot{\varepsilon}_{ij} = \begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{zx} \\ \dot{\varepsilon}_{xz} & \dot{\varepsilon}_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial V_x}{\partial x} & \frac{1}{2}\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right) & \frac{\partial V_z}{\partial z} \end{bmatrix} \tag{2}$$

A simple relationship between stress and strain rate (that we have discussed before) is given by

$$\sigma_{ij} = 2\eta\dot{\varepsilon}_{ij} \tag{3}$$

where η is the dynamic viscosity. In much of what we do, we make a statement of force equilibrium in terms of stresses, and then use the relationship between stress and strain rate, and then between strain rate and velocity.

One-Dimensional Channel Flows

The study of one-dimensional channel flows has application to flow in the asthenosphere and lower crust, as well as glaciers and mud slides. An interesting problem is, for example, the development of ductile décollements in the lower crust driven entirely by gravity (e.g., topographic slope).

A channel flow is described as one in which the velocity (or stress, etc.) does not vary in the direction of flow, only perpendicular to it. An example for a glacier is given below.

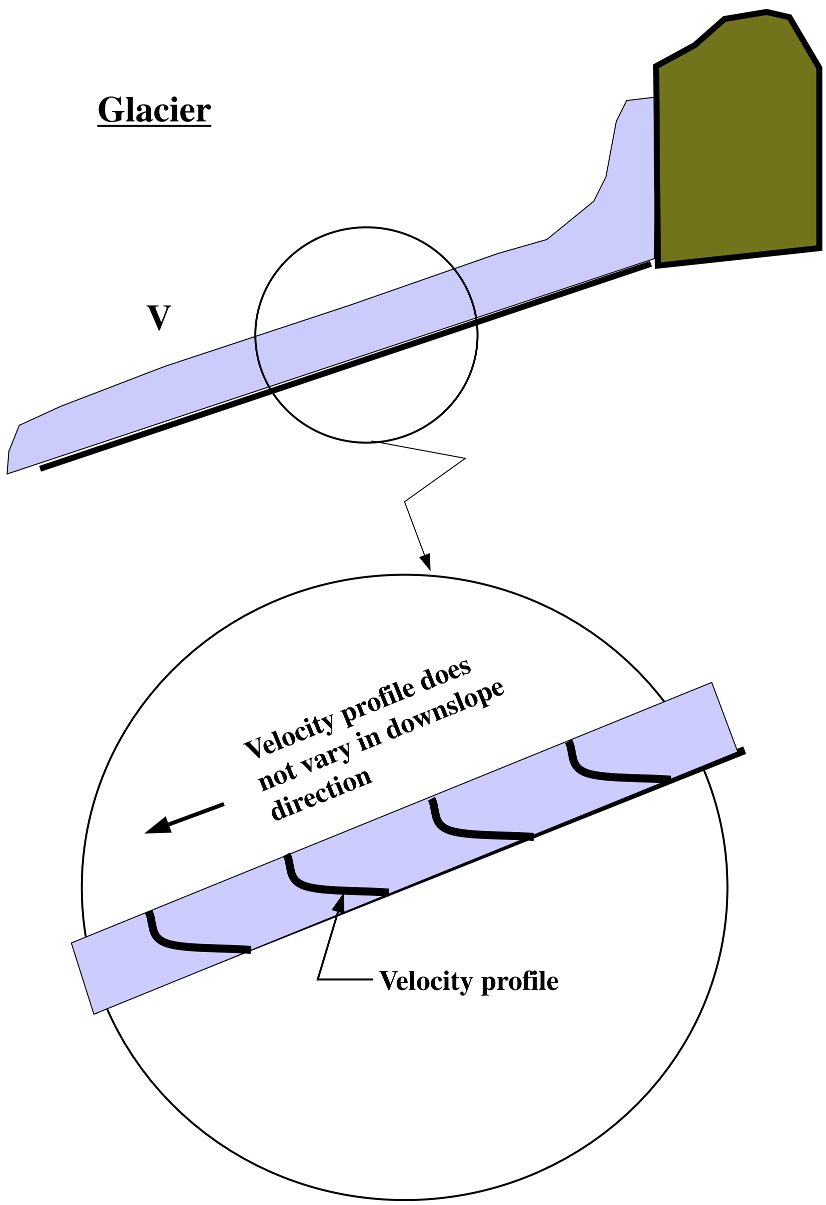


Figure 1.

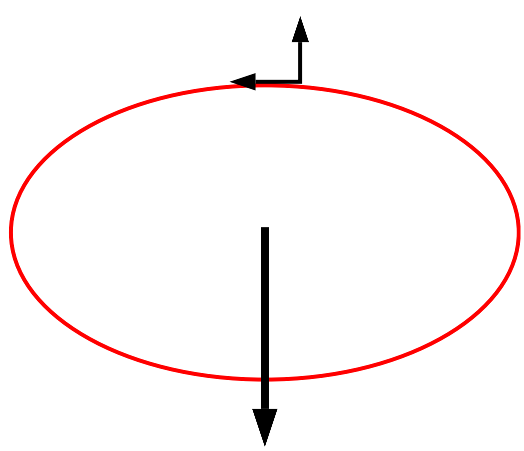
How should we approach a solution to the velocity profile in the glacier. One way is to once again consider static equilibrium, **F** = 0. Equilibrium in a fluid simply means that the velocities are steady; i.e., there are no accelerations of the fluid, hence $m\mathbf{a} = 0$. Without going through the math, it turns that in terms of stress (force per unit area) , the two-dimensional equilibrium conditions can be written as

$$
\begin{aligned}
&\mathbf{F} = 0 \quad \Rightarrow \\
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \\
&\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0
\end{aligned}
\tag{4}
$$

But stresses are not all of the forces. Stresses act on planes or surfaces (although we have to imagine every point in a body can have three infinitesimal, mutually perpendicular planes). There are also forces that act throughout a body, aptly named body forces. Body force is a vector, **f**, with components X, Y, Z, and a body force has dimensions of force per unit volume. The most common example of a body force in "earth forces" is, of course, gravity. The gravity body force only has a vertical component, usually taken to be the z axis. That is, $X = 0$, $Y = 0$, $Z = \rho g_0$. Note that we can arrive at this body force through (V is volume, $\mathbf{u_z}$ is a unit vector in the z direction))

$$
\begin{aligned}
&\mathbf{F} = m\mathbf{g}_0 = \rho V \mathbf{g}_0 = \rho V g_0 \mathbf{u_z} \\
&\mathbf{Z} = \frac{\mathbf{F}}{V} = \rho g_0 \mathbf{u_z} \equiv Z \mathbf{u_z}
\end{aligned}
\tag{5}
$$

Stresses act on surfaces



Body forces act through volume of body

Figure 2.

We can rewrite the 2-dimensional equilibrium equations for general body forces as

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{zx}}{\partial z}+X=0 \\ \frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{zz}}{\partial z}+Z=0\end{aligned} \tag{6}$$

Let us assume that the glacier has a slope with the horizontal. Now let’s set up a coordinate system with the *x*-axis in the downslope direction, and the *z*-axis of course perpendicular to the *x*-axis. Note now that in this coordinate system, the <u>gravitational</u> body force has both an *x* and a *z* component (see Figure 3):

$$\begin{aligned}X=\rho g_0 \sin\alpha \\ Z=\rho g_0 \cos\alpha\end{aligned} \tag{7}$$

<u>Glacier</u>

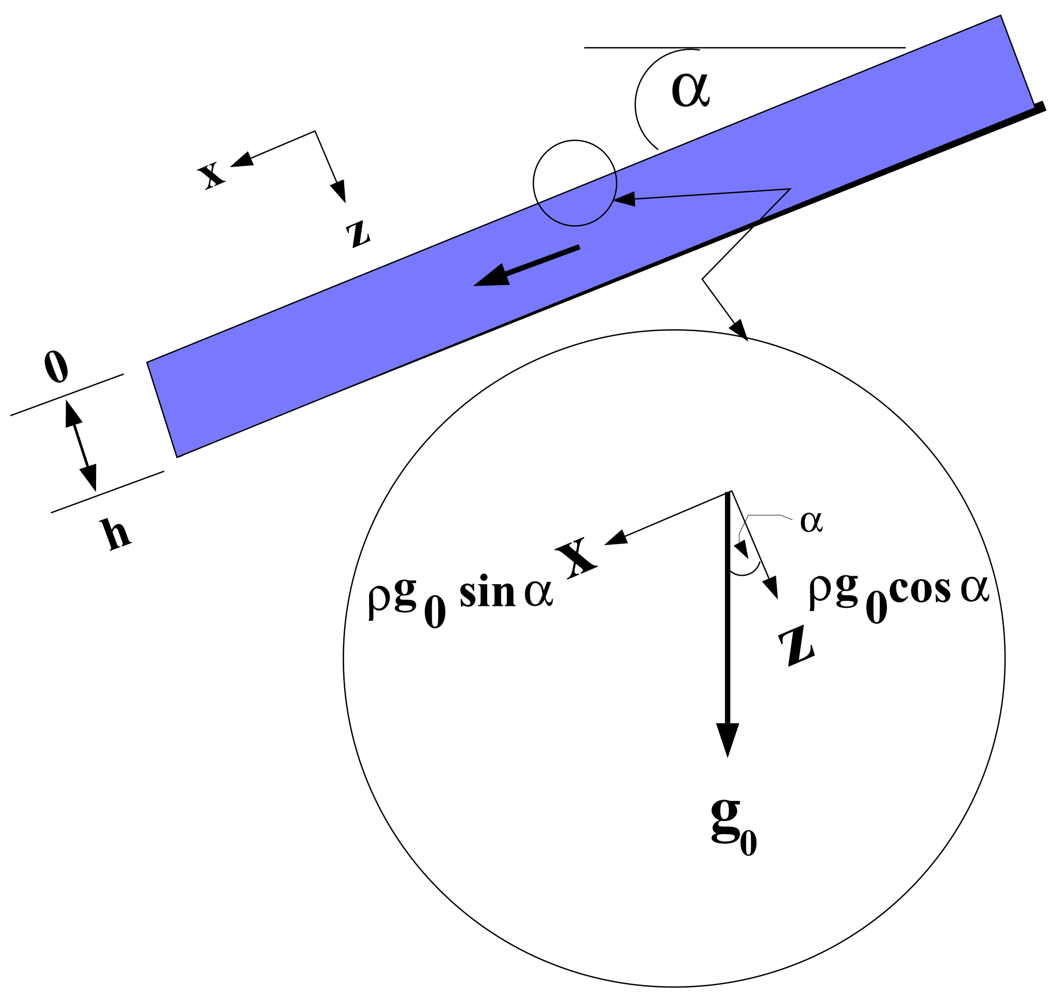


Figure 3.

We then can write the equilibrium equations as

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{zx}}{\partial z}&=-\rho g_0 \sin\alpha \\ \frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{zz}}{\partial z}&=-\rho g_0 \cos\alpha\end{aligned} \tag{8}$$

Now by our assumptions of channel flow, there are no variations in the *x* direction, i.e., $\partial / \partial x = 0$, so the first of the equilibrium equations can be written as

$$\frac{\partial \sigma_{zx}}{\partial z}=-\rho g_0 \sin\alpha \tag{9}$$

Now from equations (2) and (3), the shear stress can be related to the viscosity through

$$\sigma_{zx}=\eta\left(\frac{\partial V_x}{\partial z}+\frac{\partial V_z}{\partial x}\right) \tag{10}$$

Recognizing that the second term must be zero by our channel flow assumption, we substitute equation (10) into equation (9) to obtain

$$\eta\frac{d^2 V_x}{dz^2}=-\rho g_0 \sin\alpha \tag{11}$$

This is now a 2nd-order differential equation in the horizontal component of velocity. Note since there is a variation only in *z* that we can change a partial derivative to an ordinary derivative. Not also while there is no variation in *x*, there is certainly an *x* component to the velocity, which varies in the *z* direction.

Now the task is solve this equation subject to *boundary conditions*, of which there are two, since this is a second order differential equation. If we integrate the equation once, we get

$$\eta\frac{dV_x}{dz}=-\rho g_0 z \sin\alpha + C_1 \tag{12}$$

The surface of the glacier is at $z = 0$, and it extends downward to bedrock at $z = h$. The left hand side of equation (12) is the shear stress [see equation (10)]. At the surface there can be no shear stress, as the glacier is moving against air. Thus the left-hand-side of equation (12) must be zero at $z = 0$. The first term in the right hand side is also zero, so $C_1 = 0$. Now we integrate the equation again to get

$$\eta V_x=-\rho g_0 \frac{z^2}{2} \sin\alpha + C_2 \tag{13}$$

Our second boundary condition is that the bedrock-ice contact is so rough that the velocity is zero there:

$$\eta V_x(h) = -\rho g_0 \frac{h^2}{2}\sin\alpha + C_2 = 0 \tag{14}$$

Therefore

$$C_2 = \rho g_0 \frac{h^2}{2}\sin\alpha \tag{15}$$

Thus

$$\eta V_x(z) = -\rho g_0 \frac{z^2}{2}\sin\alpha + \rho g_0 \frac{h^2}{2}\sin\alpha \tag{16}$$

or

$$V_x(z) = \frac{\rho g_0 \sin\alpha}{2\eta}(h^2 - z^2) \tag{17}$$

This is a parabolic profile with depth in the glacier. It goes to 0 at $z = h$, and is a maximum ay $z = 0$.

Just for grins, lets assume that there is free slip at the bedrock (i.e., the shear stress $\sigma_{zx} = 0$). Then

$$\begin{aligned} \eta V_x &= -\rho g_0 \frac{z^2}{2}\sin\alpha + C_2 \\ \sigma_{zx} &= \eta \frac{dV_x(h)}{dz} = 0 = -\rho g_0 h \sin\alpha \end{aligned} \tag{18}$$

so there is no way to find out what C_2 is. **What has gone wrong here?**

When we have the solution for velocity, we can also obtain the solution for stress:

$$\sigma_{zx}(z) = 2\eta \frac{dV_x}{dz} = -2\rho g_0 z \sin\alpha \tag{19}$$

which is a maximum at $z = h$, and 0 at $z = 0$

Figure 4 below is meant to model the Saskatchewan Glacier, a tongue of ice extending about five miles eastward from the Columbia Icefield in the Canadian Rockies in Alberta. A viscosity of 10^{13} Pa s reproduces about the correct surface velocity (and $\alpha = 3°$). Ice corings confirm the parabolic behavior of the velocity with depth in the glacier.

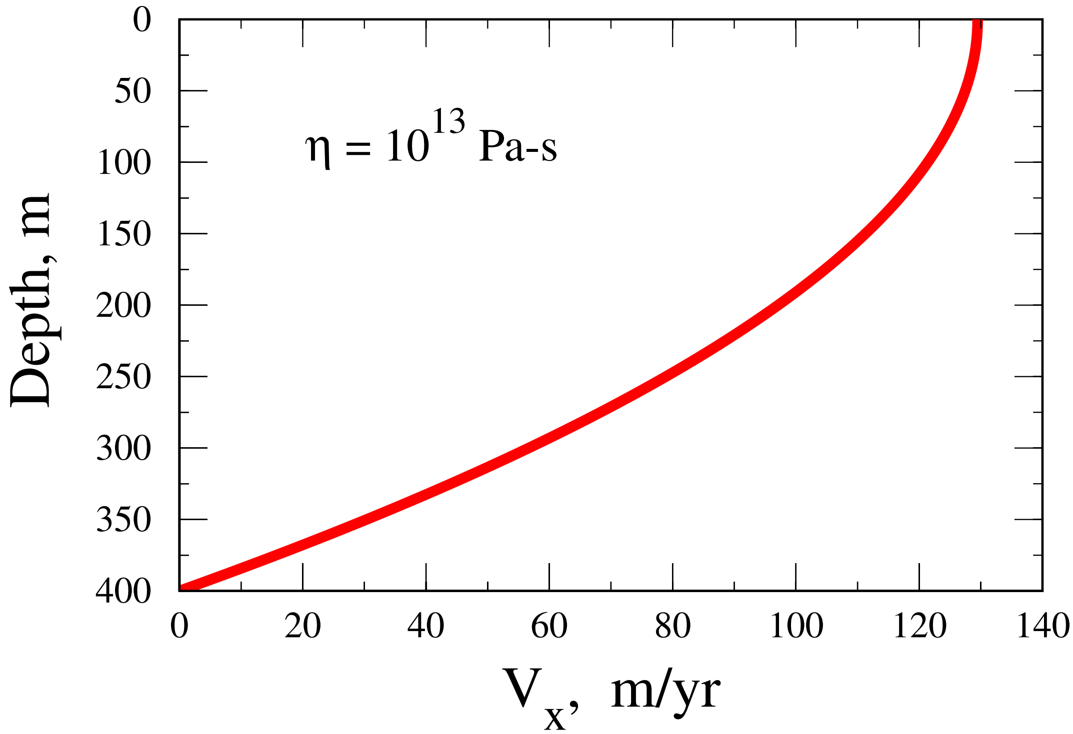


Figure 4. Velocity profile in a glacier.

We are going to take another look at a glacier, this time at the velocity transverse to the flow direction. We will assume that glacier flow is driven by a pressure gradient in the fluid (ice) and later on express the pressure gradient in terms of a gravity driving force. If we do the elemental force balance exercise (remember that we did this for the flexure problem), then it can be shown that the transverse gradient (y direction) of the horizontal shear stress must equal the downstream (x) pressure gradient:

$$\frac{d\sigma_{xy}}{dy} = \frac{dp}{dx} \tag{20}$$

The horizontal shear stress is related to velocity as (note analogy to σ_{zx})

$$\sigma_{xy} = \eta \frac{dV_x}{dy} \tag{21}$$

and substituting:

$$\eta \frac{d^2 V_x}{dy^2} = \frac{dp}{dx} \tag{22}$$

Integrating once:

$$\eta \frac{dV_x}{dy} = \frac{dp}{dx} y + C_1 \tag{23}$$

and integrating again:

$$V_x = \frac{1}{2\eta}\frac{dp}{dx}y^2 + C_1 y + C_2 \tag{24}$$

We stipulate that the velocity of the glacier goes to zero on the side walls. Thus we require that $V_x = 0$ at $y = 0$ and $y = W$, where W is the width of the glacier. The first condition requires that $C_2 = 0$. The second condition is

$$V_x(W) = 0 = \frac{1}{2\eta}\frac{dp}{dx}W^2 + C_1 W$$

This gives

$$C_1 = \frac{1}{2\eta}\frac{dp}{dx}W \tag{25}$$

so

$$V_x = \frac{1}{2\eta}\frac{dp}{dx}\left[y^2 - Wy\right] \tag{26}$$

This is the equation of a parabola, which can easily be seen if we measure from the center of the glacier by defining a new variable $y' = y - W/2$:

$$V_x = \frac{1}{2\eta}\frac{dp}{dx}\left[y'^2 - \frac{W^2}{4}\right] \tag{27}$$

How can we relate the pressure gradient to gravity? Consider two positions (1 and 2) on the glacier a distance Δx apart in the down slope direction (see Figure 5).

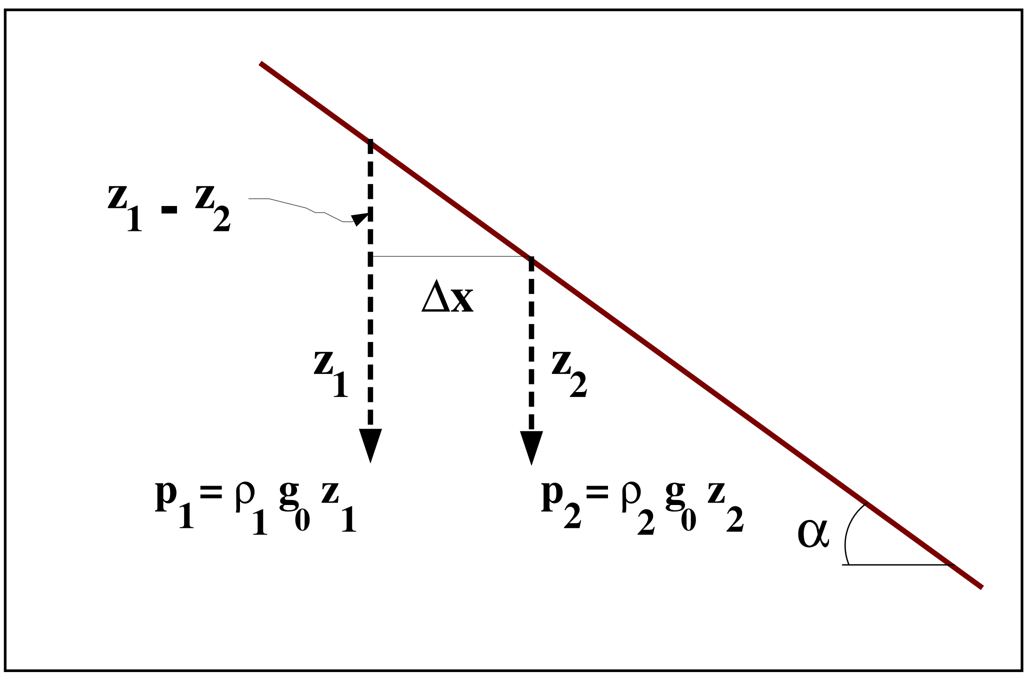


Figure 5.

At the same level in the ice, the two pressures are $p_1 = \rho_{ice} g_0 z_1$ and $p_2 = \rho_{ice} g_0 z_2$.
The difference in pressure is

$$\Delta p = p_2 - p_1 = -\rho_{ice} g_0 (z_1 - z_2) \tag{28}$$

But by trigonometry

$$(z_1 - z_2) = \Delta x \tan \alpha \approx \Delta x \alpha \tag{29}$$

where the approximation is possible for small values of α. Thus

$$\Delta p = -\rho_{ice} g_0 \alpha \Delta x \tag{30}$$

and in the limit the pressure gradient is

$$\frac{dp}{dx} = -\rho_{ice} g_0 \alpha \tag{31}$$

so

$$V_x = \frac{\rho_{ice} g_0 \alpha}{2\eta} \left[\frac{W^2}{4} - y'^2 \right] \tag{32}$$

A solution for $= 10^{14}$ Pa s and $\alpha = 3°$ is shown in Figure 6

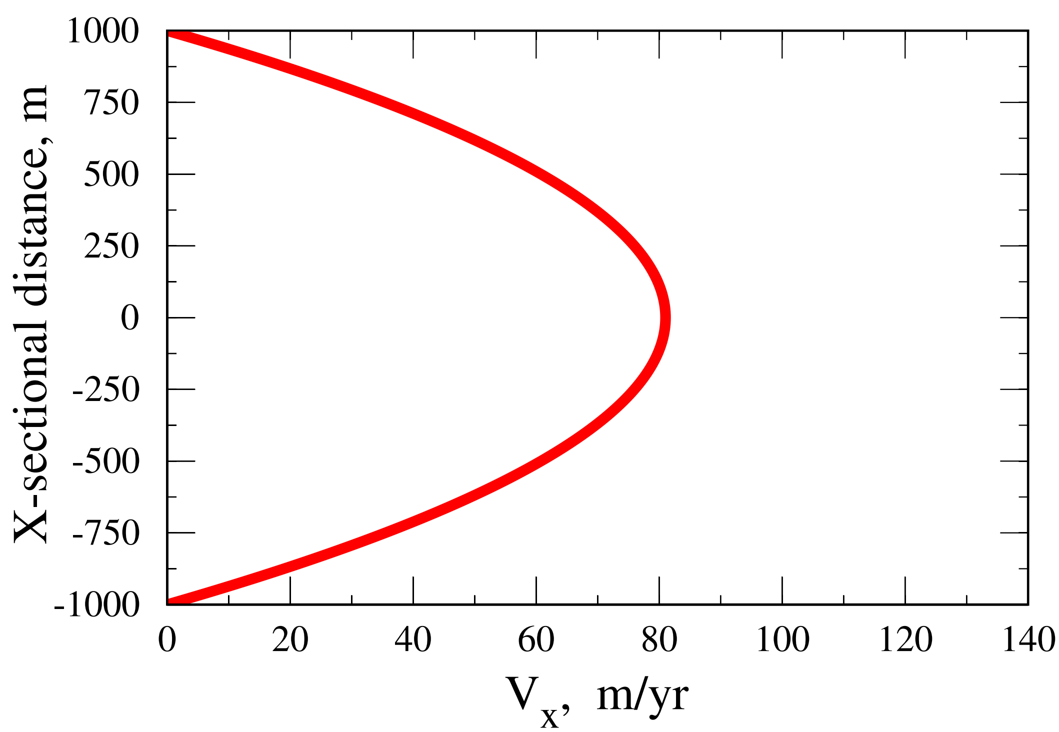


Figure 6.

The figures in the PowerPoint file show that the models for both the vertical and transverse velocities work well for the Saskatchewan glacier.

Note that the considerations here apply to any viscous flow: lava, mud, etc.