Flexure – Part II

See Chapter 6.2.5 of Lowrie

Solution to the Flexure Equation for Subduction

We will examine in detail the solution to the flexure equation for subduction of the oceanic lithosphere. That is, what shape does the lithosphere take as it bends into a trench? In this case, there is no in-plane force (P = 0) and we write the general thin-plate equation as:

$$D\frac{d^4 w(x)}{dx^4} + \Delta\rho g_0 w(x) = q_a(x)$$

Note that when $D \rightarrow 0$ (i.e., when Young's modulus is very small, or when the plate is very thin) or when $dw/dx \rightarrow 0$ (wavelength is very long), then

$$\Delta\rho g_0 w(x) = q_a(x)$$

If the load is due to topography $h(x)$, then

$$q_a(x) = \rho_c g_0 h(x)$$

and because $\Delta\rho = \rho_m - \rho_c$ [the crust (c)-mantle (m) density contrast], this leads to

$$w(x) = \frac{\rho_c h(x)}{(\rho_m - \rho_c)}$$

which is **ISOSTASY!** The isostasy can be viewed as a special case of flexure when $D \rightarrow 0$.

When the load is zero, this is a homogeneous 4th order ordinary differential equation. How do you solve it? The general solution is:

$$w(x) = [A_1 \sin(k_0 x) + A_2 \cos(k_0 x)] \exp(-k_0 x)$$
$$+ [A_3 \sin(k_0 x) + A_4 \cos(k_0 x)] \exp(+k_0 x)$$

where A_i, i = 1, 2, 3, 4 are constants to be determined from *boundary conditions*, and

$$k_0 = \left[\frac{g_0 \Delta\rho}{4D} \right]^{1/4}$$

is the *flexural wavenumber* of the lithosphere. Its reciprocal is often designated by α and termed the *flexural parameter* of the lithosphere. This gives an indication of how far away from its loading source will lithospheric stress and deformation be important. Alternatively, it is the lateral extent of regional compensation of a concentrated load. For typical oceanic lithosphere, D = 5x 10^{21} N m and $\Delta\rho$ = 2000 kg m^{-3}, and the flexural parameter is 30 km. Thus a typical result of loading may show up as damped, single cycle sinusoid.

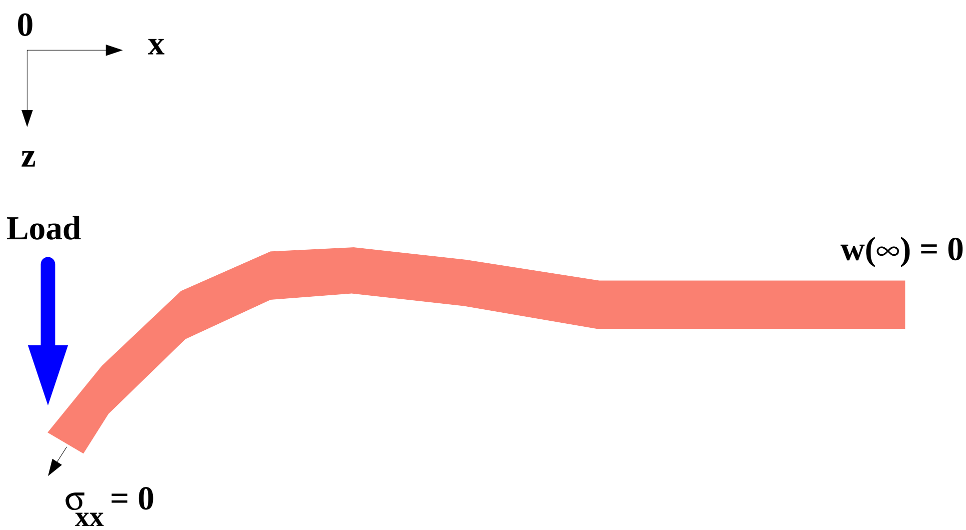

Figure 6. Flexure of the lithosphere near a subduction zone

In order to use the general solution, four boundary conditions need to be specified because there are 4 unknown constants in the solution of the differential equation. We examine here the bending of the lithosphere at a subduction zone by a concentrated load in the trench. This load might represent the weight of the subducted slab. We set x = 0 in the trench (and apply the load there; x positive is seaward).

Since

$$w(\infty) = 0, \quad \text{then} \quad A_3 = A_4 = 0$$

Stress is not transmitted across the blunt end of the plate, so that

$$\sigma_{xx}(0) = 0 \Rightarrow \frac{d^2 w(0)}{dx^2} = 0$$

Taking the second derivative, we find a cosine term associated with A_1, so it must be zero. We have one more constant to find. It is a little tricky, and we just give the result below that

$$\frac{d^3 w(x)}{dx^3} = \frac{q_a^\$}{D}$$

where $q_a^\$$ can be considered as the average stress imparted by the concentrated load times the distance the load is distributed along the plate in the x direction (thus $q_a^\$$ has the dimensions of Pa m). The load is still concentrated at the end of the plate, we are just saying that it has a finite width. Calculating the third derivative of our solution and setting it to $q_a^\$ D^{-1}$ yields

$$A_2 = \frac{q_a^\$}{2Dk_0^3}$$

so that our solution for the subduction problem is:

$$w(x) = \frac{q_a^\$}{2Dk_0^3} \cos(k_0 x) \exp(-k_0 x)$$

and the stress follows from the relationship between stress and curvature (d^2w/dx^2) as

$$\sigma_{xx}(x) = -\frac{E}{(1-\nu^2)} \frac{q_a^\$ z}{2Dk_0} \sin(k_0 x) \exp(-k_0 x)$$

In Figure 7 below we plot solutions for $w(x)$ and $\sigma_{xx}(x)$ where $q_a^\$ D^{-1}$ has been set so that the multiplying constant to each solution is unity. The load at the origin causes the lithosphere to be depressed into a trench between $x = 0$ and $x = \lambda_0/4$, where λ_0 is the *flexural wavelength* ($\lambda_0 = 2\pi/k_0$). The prominent uplifting of the lithosphere is called the *flexural bulge* or *flexural forebulge*. In subduction parlance, this is the *outer rise*. High tensional stresses occur in the upper half of the plate between the trench and the flexural bulge. These give rise to normal faulting and earthquakes.

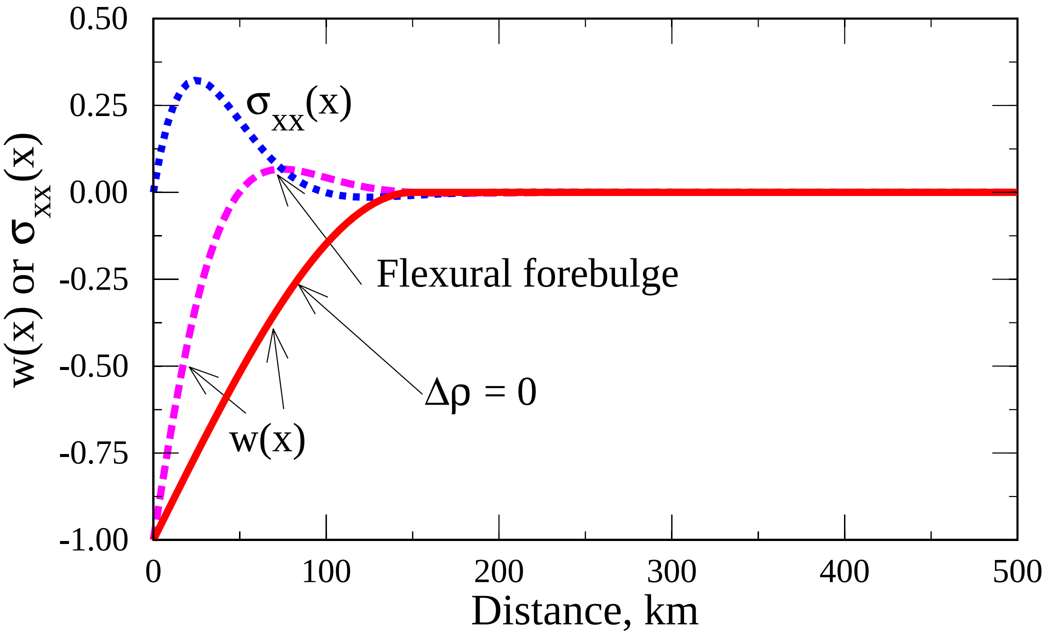

Figure 7. $D = 5\text{x}10^{21}$ N m, $\Delta\rho = 2000$ kg m^{-3}, and $g_0 = 10$ m s^{-2}.

How can the flexural rigidity estimated from fitting a flexural curve to a subduction zone profile be used to estimate temperature in the lithosphere?

An example of flexure on Venus is shown in Figure 8 below. The figure shows the topography (shaded) in a region alleged by some to be a subduction zone. The dashed line is a best-fitting flexural model.

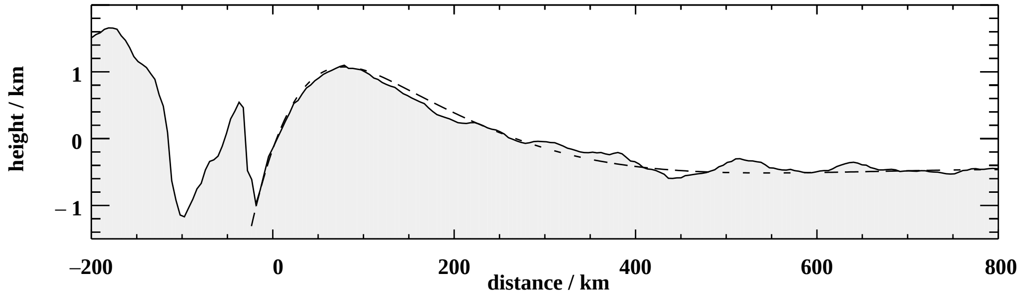

Figure 8

Subduction zone flexure with no density contrast

In our "subduction flexure" lab it will not be possible to have a density contrast between what is below the plate (air) and what is above the plate (air). Therefore our flexure equation will become:

$$\frac{d^4w(x)}{dx^4} = \frac{q_a(x)}{D}$$

A simple way to approach this problem is to consider a plate of length L pinned at $x = 0$ [$w(0) = 0$], where the slope of the plate is also zero [$dw(0)/dx = 0$]. If a load $q_a^\$$ is applied at $x = L$, then it is obvious that the moment is given by

$$M(x) = q_a^\$(x - L)$$

But we know that

$$M(x) = -D\frac{d^2w(x)}{dx^2}$$

so

$$\frac{d^2w(x)}{dx^2} = -\frac{q_a^\$}{D}(x - L)$$

If we integrate this equation once, we have

$$\frac{dw(x)}{dx} = -\frac{q_a^\$}{D}\left(\frac{x^2}{2} - Lx\right) + C_1$$

but the constant C_1 must be zero because $dw(0)/dx = 0$. Integrating again:

$$w(x) = -\frac{q_a^\$}{D}\left(\frac{x^3}{6} - L\frac{x^2}{2}\right)$$

or

$$w(x) = \frac{q_a^\$x^2}{2D}\left(L - \frac{x}{3}\right)$$

where again the constant of integration must be zero because $w(0) = 0$.

The lab setup and a plot of a solution are shown below. The objective of the lab exercise is to estimate the flexural rigidity of the "plate".

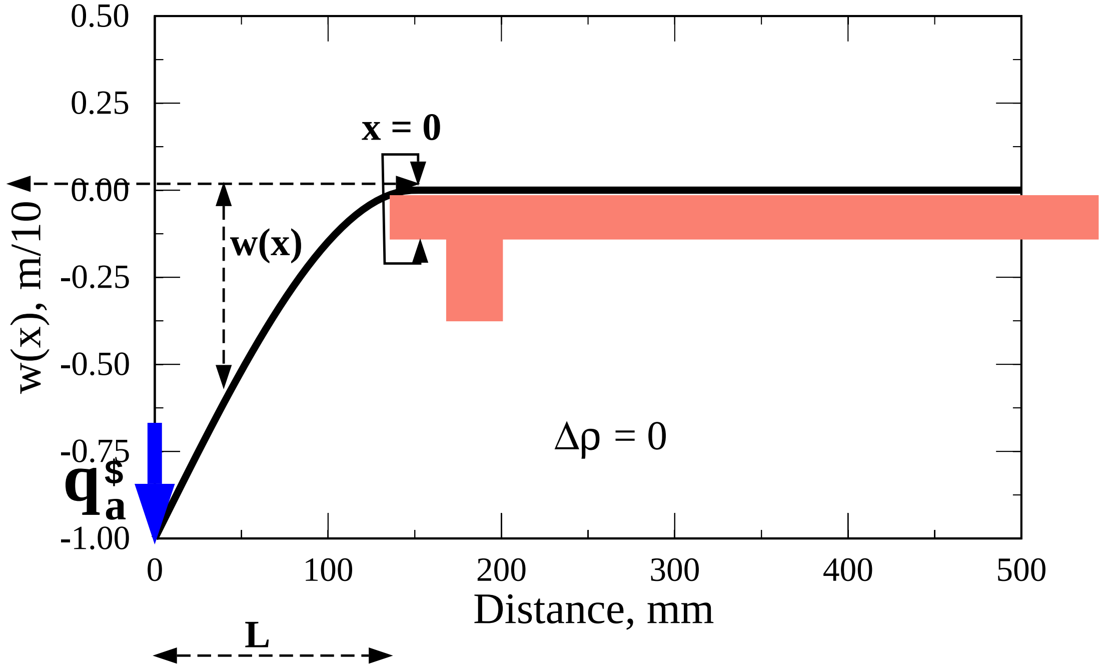

Figure 9. Lab setup.