Flexure – Part I

See Chapter 6.2.5 of Lowrie

Introduction

There is a long tradition in treating the Earth's lithosphere as a thin elastic plate, and treating its deformation as flexure of a beam. Such an approximation works as long as the wavelengths involved are large compared to the plate thickness. By “wavelength”, we mean the wavelength of an applied load on the plate, either from above or below. With the “thin-plate approximation”, the equations describing this problem are greatly simplified, as are the relationships for stresses and displacements.

If we confine our interest to wavelengths between a few hundred and a few thousand kilometers, the Earth's lithosphere can be modeled as a homogeneous elastic layer supported by a very weak substratum called the *asthenosphere*. The asthenosphere behaves like a fluid since it cannot support shear stresses for long periods of time. Since typical lithosphere strains are small, the main deformation of the lithosphere consists of up and downwarping, or *flexure* of the layer as a whole. This flexure can be quantified by the vertical distance, $w(x)$, that the middle(or *median surface*) of the layer has been deflected from its reference position. Here we will consider only one horizontal dimension with deflection . Further, if we assume that the deformation is small enough such that straight lines originally perpendicular to the median surface remain straight and perpendicular to it during the deformation, then the flexure completely describes the state of stress in the lithosphere. This implies that $dw(x)/dz = 0$.

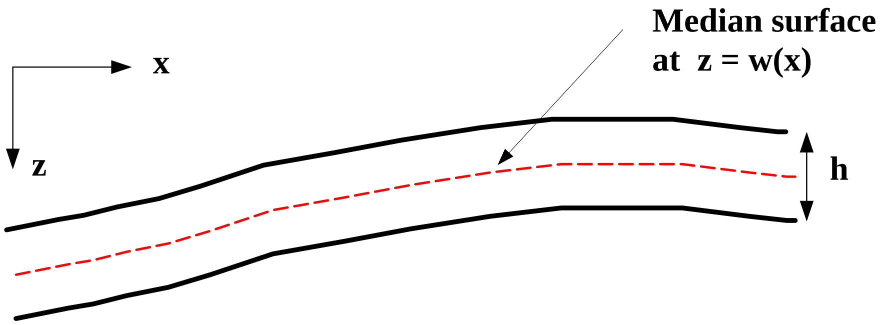


Figure 1

Static Equilibrium

We can apply simple concepts of Earth Forces, in this case force and moment balance, to learn about stresses in a flexed plate. The approach here is one of static equilibrium. In Newton's second law, **F** = *m***a**; but there are no accelerations, so we demand situations in which all the forces, **F**, sum to zero.

We consider the Cartesian system (*x*, *y*, *z*), with *y* along the strike, or two-dimensional axis, of the problem. The deflection *w*(*x*) is specified as δz.

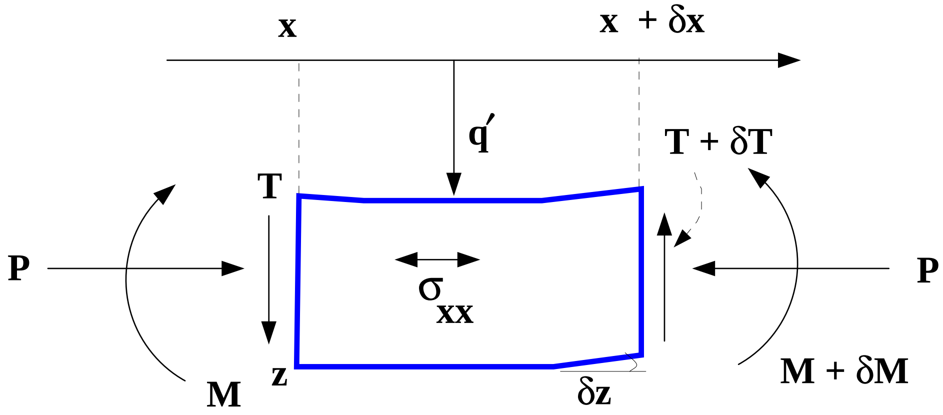


Figure 2. The *y* direction is into the plane of the paper.

We balance forces and moments on a small section or element of a lithospheric plate that is undergoing bending. First we examine force balance. An external load, *q'*(*x*), is applied to the element. (The *x*-dependency is understood, so we just designate the load as *q'*.) The load has dimensions of force per unit distance in the *y* direction per unit distance in the *x* direction, or N m^{-2} (= Pa, or the units of stress). The force acting on the element per unit distance in the *y* direction is thus $q'\delta x$ and loads the plate. The force is resisted by shear forces on the vertical boundaries of *T* and $T + \delta T$ (forces per unit distance in the *y* direction). The net forces must balance to zero.

$$q'\delta x + T - (T + \delta T) = 0$$
$$q'\delta x - \delta T = 0$$

In the limit for small quantities, δx approaches *dx*, and

$$\frac{dT}{dx} = q'$$

We do not consider shear forces on the top and bottom surfaces; they are considered to be shear-free (atmosphere, “asthenosphere”).

Not only do we have to balance the forces, we have to balance any *moments* acting on the element. A moment is a force times a moment arm, and tends to cause a torque or rotation. There can be no net rotation on the element in static equilibrium. To consider moment balance about the origin, we first consider the force acting on the vertical sides of the element. The moment due to shear forces is

$$Tx - (T + \delta T)(x + \delta x) \approx -T\delta x$$

Note that the minus sign follows because the positive z direction is downward. Note also that the change in T is small in going from x to $x + \delta x$, so the term is neglected, as is $\delta x \delta T$ because it is the product of small perturbations.

The horizontal forces per unit distance in the y direction, P, acting on the two ends, though equal and opposite, act at different levels, so that the net moment is

$$Pz - P(z - \delta z) = P\delta z$$

So far we have just considered force balance on the boundaries of the plate. But as the plate bends, internal stresses also give rise to a *bending moment*. The bending moment acting on the element being considered is

$$M - (M + \delta M) = -\delta M$$

This moment is associated with a *bending* or *fiber stress*, σ_{xx}. In a plate of thickness h, the moment is just the integral of stress (acting in the x direction) times the moment arm (acting in the z direction). A moment is force × distance. Then the moment is stress (force/distance^2) × distance × distance, or force × distance per unit distance in the y direction. Thus moment has the dimension of force (Newtons). It is given by:

$$M = \int_{-h/2}^{h/2} \sigma_{xx} z dz$$

and

$$M + \delta M = \int_{-h/2}^{h/2} (\sigma_{xx} + \delta\sigma_{xx}) z dz$$

The net moment is

$$M + \delta M - M = \delta M = \int_{-h/2}^{h/2} \delta\sigma_{xx} z dz$$

where the midplane (or *neutral plane*) is at $z = 0$; it is convenient to calculate the moment form the neutral plane, for here $\sigma_{xx} = 0$ (**Why is that?**). We do not need to consider the moments provided by $q'(x)$ because we assume that it is of a sufficient wavelength (compared to the thickness of the plate) such that it does not exert significant torque on the slice $[q'(x) \approx q'(x + \delta x)]$. (The integral of $q'(x)x$ vanishes). This is inherent in the thin-plate approximation. Taking the limit as δx and δz approach zero, momentum balance requires

$$-dM - Tdx + Pdz = 0$$

or

$$\frac{dM}{dx} - P\frac{dz}{dx} = -T$$

Now differentiate this equation with respect to x and use the terminology for the deflection $w(x) = z$:

$$\frac{d^2M}{dx^2} - P\frac{d^2w}{dx^2} = -q'$$

where we have substituted $q = dT/dx$ and

$$M = \int_{-h/2}^{h/2} \sigma_{xx} z dz$$

So we have arrived at the equation of static equilibrium for a infinitesimal element subject to an external load and a horizontal force. Note that equation has nothing to do with the rheology of the plate; it could be viscous, plastic, or elastic.

Let's examine the dimensions of this equation. Bending moment is stress times distance integrated over distance, so it must be in N, as noted above. Any differential operator, "d/d-anything" has units of "anything^{-1}", so the first term is in units of N m^{-2}. In the second term, P is a force per unit distance and is best imagined as a horizontal pressure in the plate integrated over the thickness of the plate (this is often called the "*in-plane force*"). The derivative has units of m^{-1}; so this term is also N m^{-2}. The term on the right hand side is in N m^{-2}, as originally stated. This is a good check: the dimensionality of every term in a physical equation should be the same.

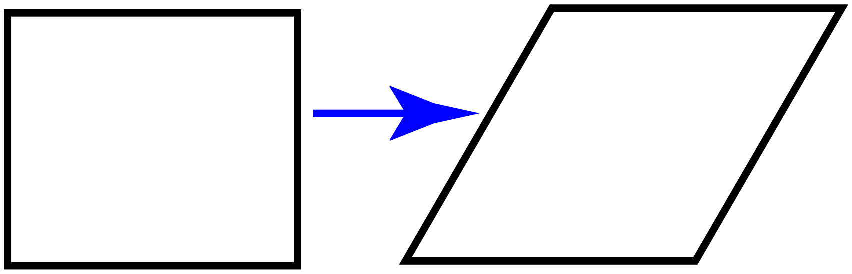
Relating Fiber Strain, ε_{xx}, to Displacement

So we have this equation of equilibrium

$$\frac{d^2M}{dx^2} - P\frac{d^2w}{dx^2} = -q'$$

Presumably we know the in-plane force, P, and the load q' "going in". To get any further, we have to relate vertical displacement, $w(x)$ to bending moment, $M(x)$, so that we have an equation in a single unknown variable. The key is to use the thin-plate approximation and relate $w(x)$ to *fiber strain*, ε_{xx}. Consider a plate as a "bundle" of fibers in the x direction. When the plate is loaded from the top and its ends are supported, it is bent downward, and the top of the plate is in compression, the bottom in extension. The fibers are shortened at the top of the plate and elongated at the bottom of the plate.

The concept of strain has already been discussed. A simple definition is that is the change in shape of an object relative to its original shape. Strain at a point has the same terminology as stress. There are a total of 9 strain components and on any of 3 mutually perpendicular planes there are two shear strains and one normal strain. Normal strains refer to simple elongations or contractions (e.g., If you heat a rod it will lengthen according to the coefficient of thermal expansion. The new length divided by the old length is the normal strain: $\varepsilon_x = (x + \Delta x)/x$. Shear strains refer to a change in dimension of an object per unit change in distance in a perpendicular direction; it is best visualized as the change in angle between two lines in an object that were initially perpendicular. In our bending moment problem we are only interested in normal strain in the x direction acting in the x plane, or ε_{xx}.



Simple Shear

Figure 3

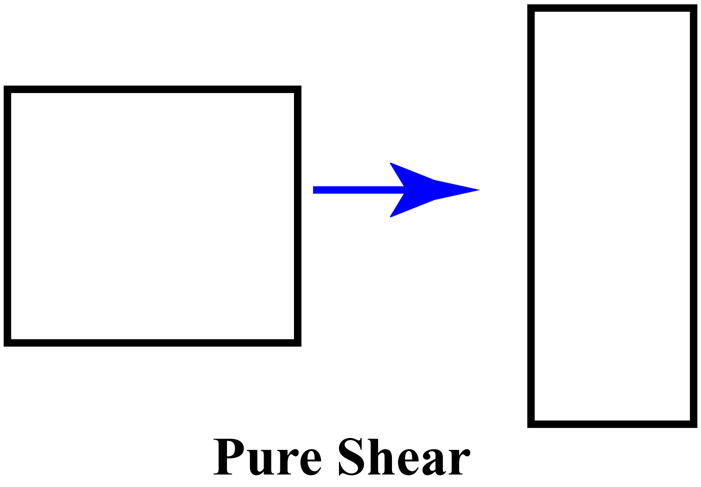


Figure 4

We are going to assume that the plate is thin, so that every horizontal surface bends exactly the same way. Thus, there is no vertical strain, ε_{zz}. So this is even simpler than pure shear, it is 1-dimensional strain. Now it follows that the more the plate is bent, the more the fibers are stretched. A measure of the bending is the curvature of the plate, or $d^2w(x)/dx^2$.

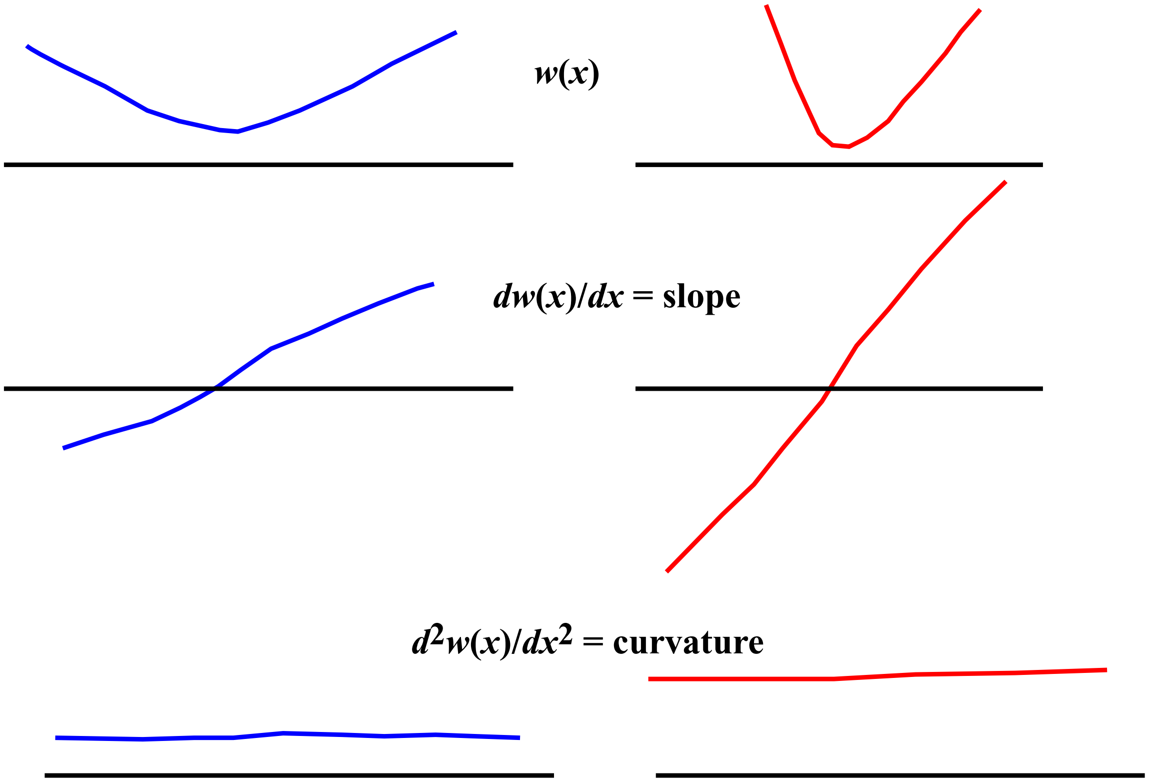


Figure 5. Two examples of curvature.

So the strain must be proportional to the curvature. Consider a plate loaded from above. The top part of the plate will be in maximum compression and the bottom part of the plate will be in maximum extension. The strain must pass through zero in the middle of the plate at $z = 0$. We appeal to your intuition that the following represents the strain in the bent plate (x dependence implicit):

$$\varepsilon_{xx} = -z\frac{d^2w}{dx^2}$$

The minus sign comes about because we take compression as negative and z is positive upwards.

So far we have said nothing about the rheology of the plate. We will now assume that it is elastic. Stress and strain in an elastic plate are related through an elastic constant. In general, stress = *Young's modulus* (E) times strain. E is just a measure of how much an elastic material will strain under an applied stress. One other property is necessary to describe an elastic material, *Poisson's ratio*, ν. Poisson's ratio is just a measure of the perpendicular strain of an elastic material. If you take a rod and extend it elastically, its cross-sectional diameter must decrease. This perpendicular strain is related to longitudinal strain by ν. Because we have assumed that there is no vertical strain, it turns out that fiber stress is related to fiber strain through:

$$\begin{aligned}\sigma_{xx}(x) &= \frac{E}{(1-\nu^2)}\varepsilon_{xx}(x)\\ &= -\frac{zE}{(1-\nu^2)}\frac{d^2w(x)}{dx^2}\end{aligned}$$

We can now relate moment to curvature:

$$\begin{aligned}M(x) &= \int_{-h/2}^{h/2}\sigma_{xx}(x)z\,dz = -\frac{E}{(1-\nu^2)}\frac{d^2w(x)}{dx^2}\int_{-h/2}^{h/2}z^2dz\\ &= -D\frac{d^2w(x)}{dx^2}\end{aligned}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

is the ***flexural rigidity*** of the plate.

We can now substitute this relationship in to the equation of equilibrium to obtain

$$D\frac{d^4w(x)}{dx^4}+P\frac{d^2w(x)}{dx^2}=q'(x)$$

Now it turns out that if you push down with a load, $q_a(x)$, and displace material below the plate of density ρ_{below} with material above the plate of density ρ_{above}, the displacement will push back on the load (if $\rho_{\text{below}} > \rho_{\text{above}}$) with a restoring force

$$(\rho_{\text{below}}-\rho_{\text{above}})g_0w(x)\equiv\Delta\rho g_0w(x)$$

This is nothing more than Archimdes Principle. The effective load is then

$$q'(x)=q_a(x)-\Delta\rho g_0w(x)$$

so that the flexure equation becomes:

$$D\frac{d^4w(x)}{dx^4}+P\frac{d^2w(x)}{dx^2}+\Delta\rho g_0w(x)=q_a(x)$$

This is arguably the single most important equation for studying the geodynamical deformation of the lithosphere in response to loads and in-plane forces.