Spherical Earth mode and synthetic seismogram computation

Guy Masters

CIG-2006

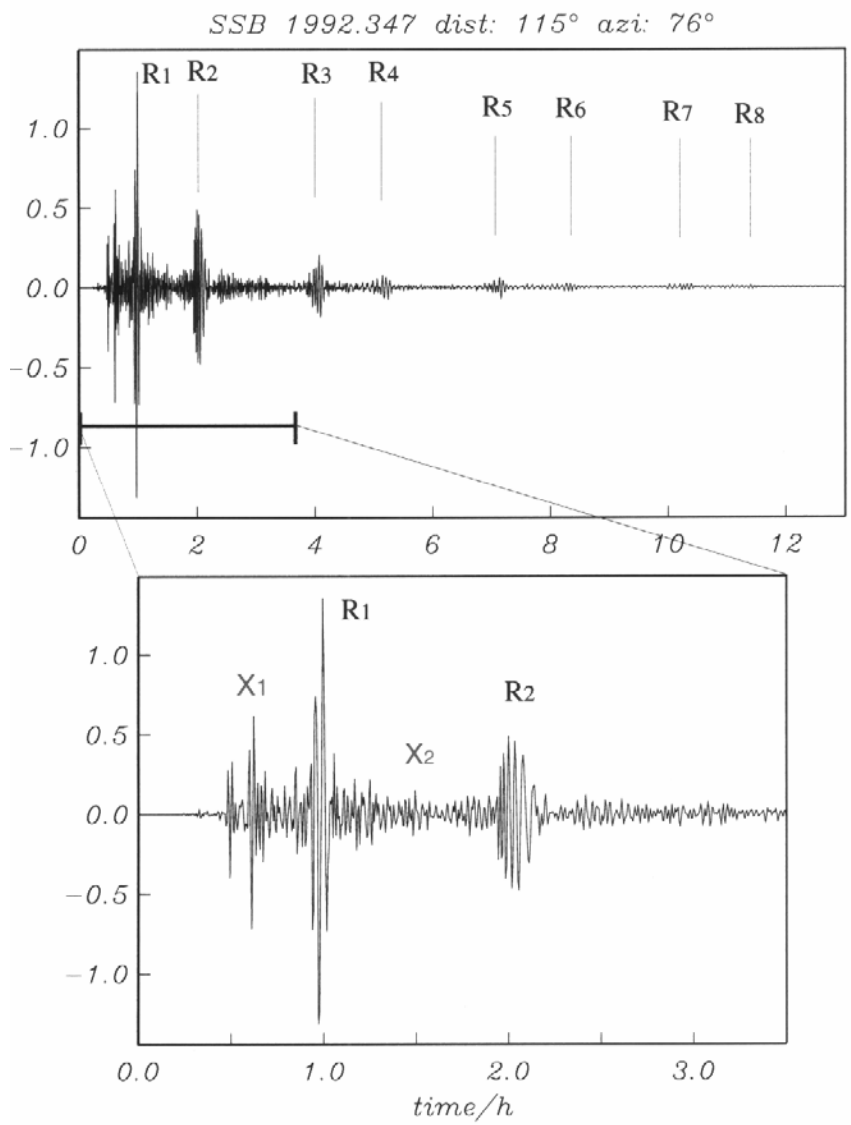
MINEOS code package

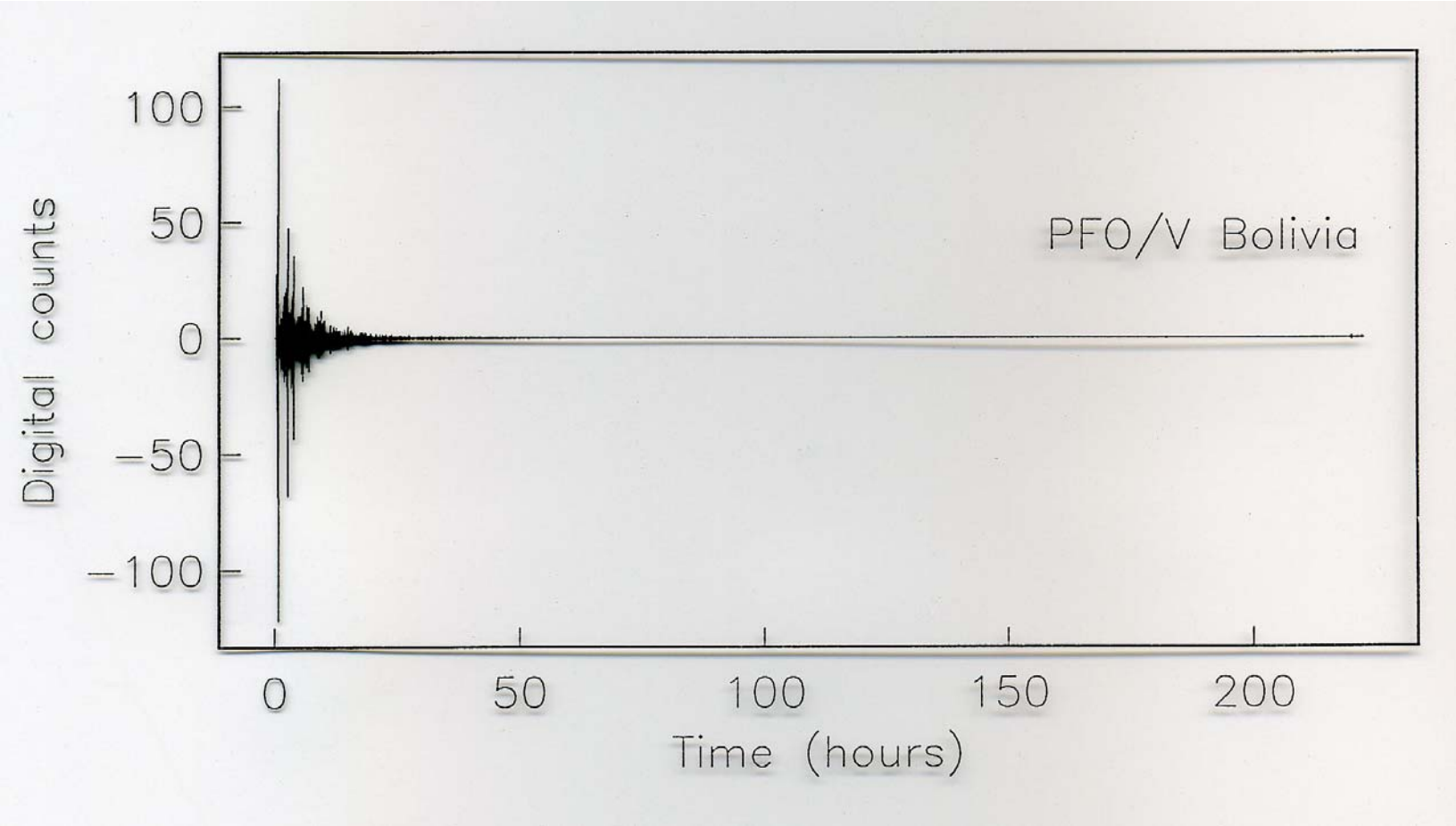
- mineos_bran: does mode eigenfrequency and eigenfunction calculation -- this is where all the work is!
- eigcon: massages eigenfunctions for greens function calculation
- green: computes greens functions for a point source
- syndat: makes synthetics for double-couple or moment tensor sources

Algorithm history

- Variable order, variable step-size Runge-Kutta (up to 8'th order) (E.B. Shanks, 1966, Math. Comp, 21–38). Tests show superior to Burlisch-Stoer depite what Numerical Recipes says. Original coding by Freeman Gilbert, circa 1966 (program EOS)
- Variable step size based on growth rate estimated from estimates of the eigenvalues of the matrix $\mathbf{A}$ (Gilbert, Freeman and G. Backus, 1969 Proc. of the Fall Joint Computer Conference, 1968, of the Am. Federation of Information Processing Soc.)
- Solution for spheroidal modes is computationally unstable – use method of minors (Gilbert and Backus, 1966, Geophysics, 31, 326–332). This was implemented by John Woodhouse circa 1980 with a clever method of computing eigenfunctions.
- Introduce mode counter: originally for spheroidals by John Woodhouse. I added counters for toroidals and radials
- Codes benchmarked against Rayleigh-Ritz code of Ray Buland circa 1981. Slight differences tracked down to use of a slightly different value of the gravitational constant??
- Code added to compute accurate eigenfunctions of "difficult" modes (e.g. Stoneley and IC modes) – circa 1985?? (I don't remember!)

Some background on modes

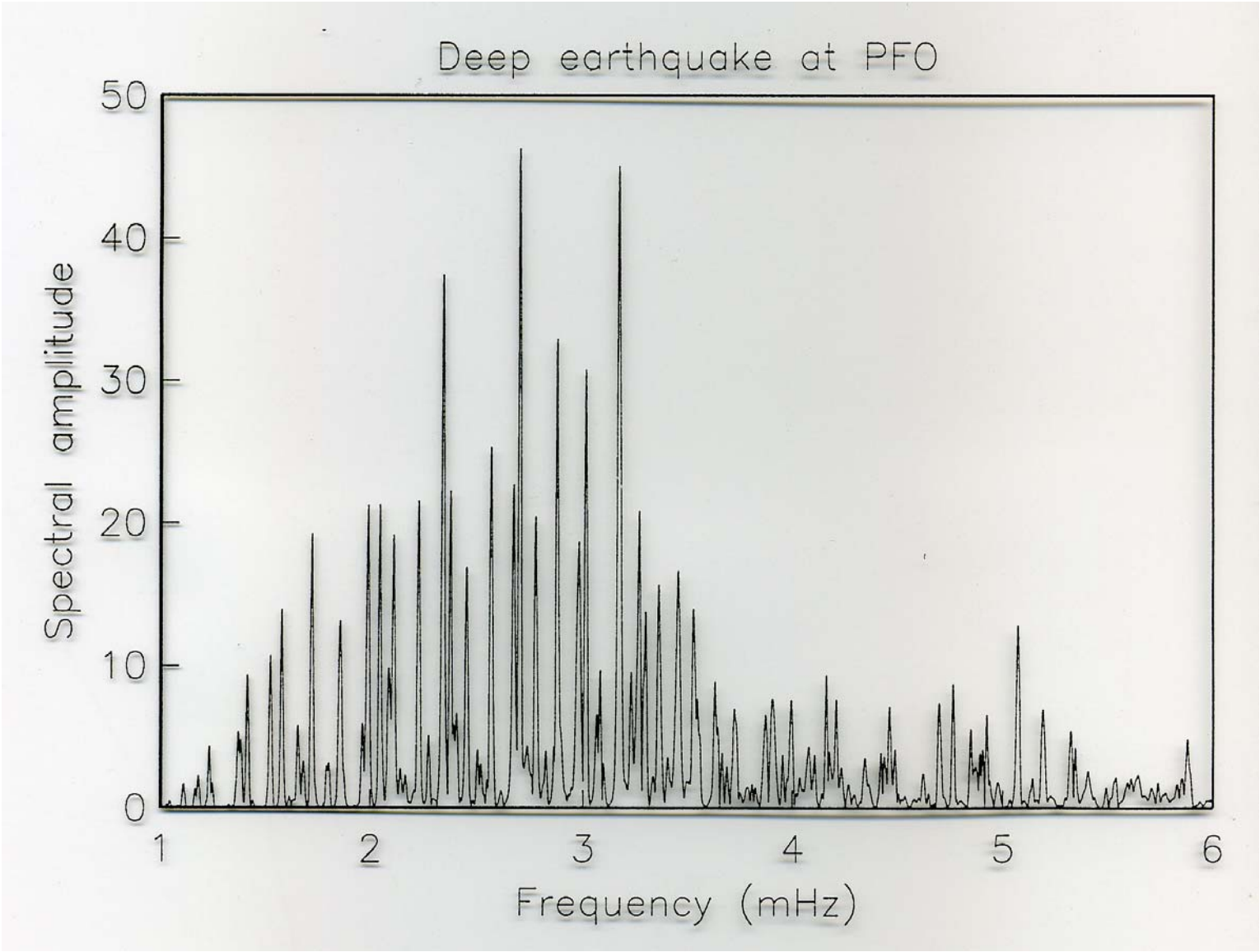

SSB 1992.347 dist: 115° azi: 76°
R1
R2
R3
R4
R5
R6
R7
R8
1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
12
R1
X1
X2
R2
1.0
0.5
0.0
−0.5
−1.0
0.0
1.0
2.0
3.0
time/h

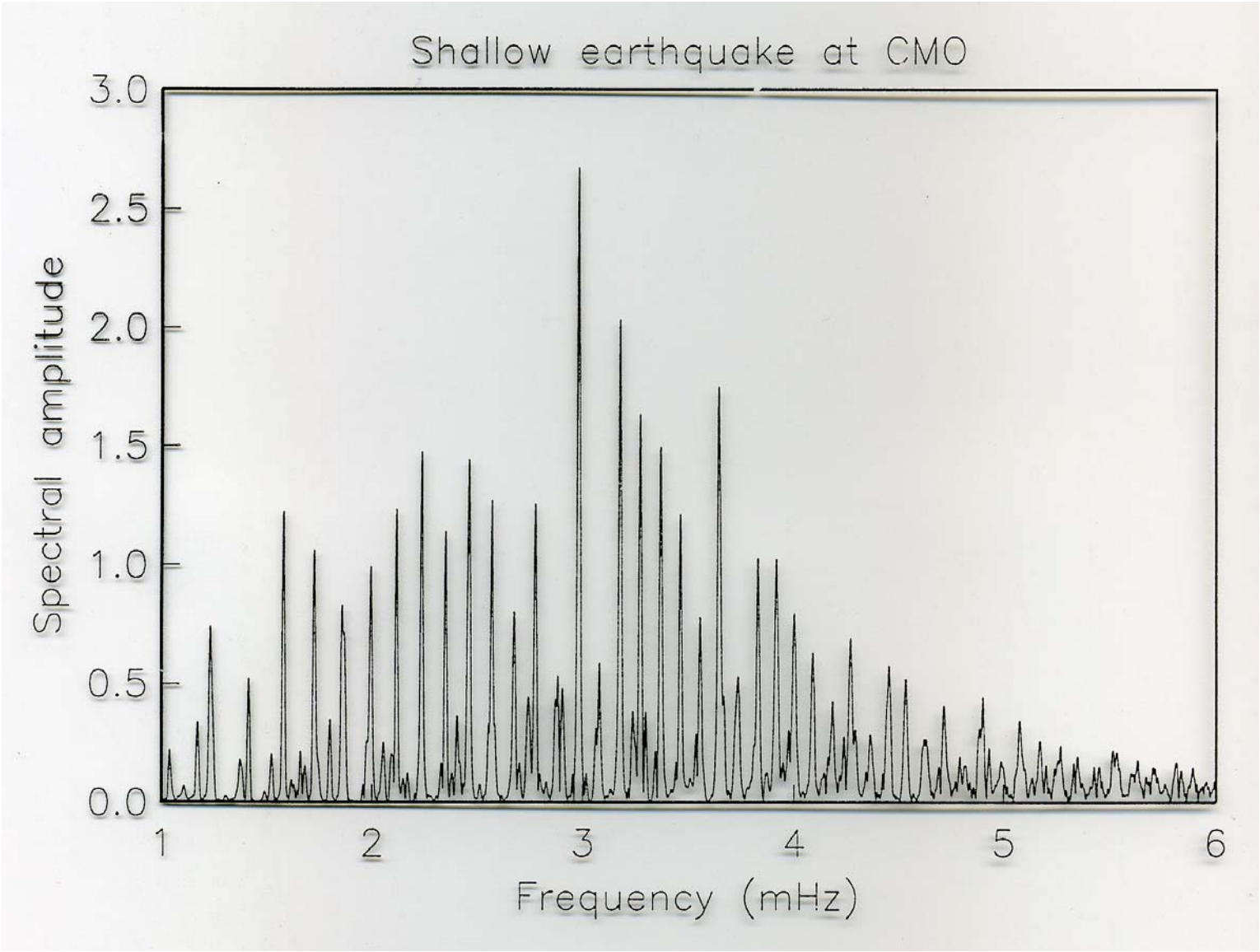

PFO/V Bolivia
Digital counts
100
50
0
−50
−100
0
50
100
150
200
Time (hours)

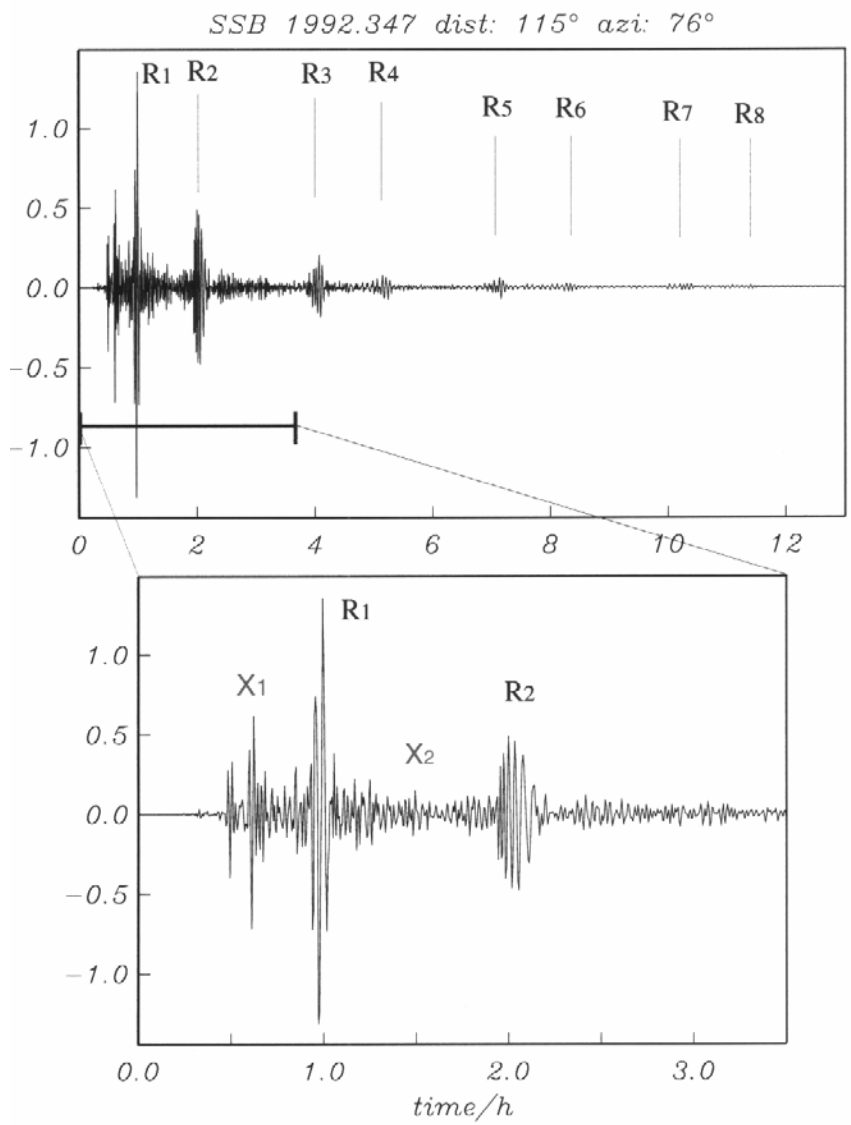
Seismogram is a sum of decaying cosinusoids

$$s(t) = \sum_k A_k \cos(\omega_k t + \phi_k) e^{-\alpha_k t}$$

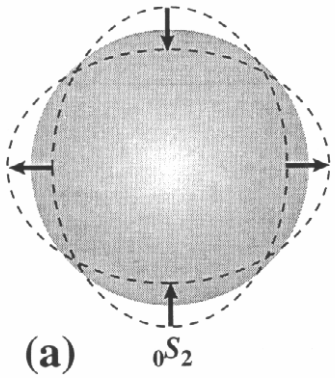
ω_k, α_k are the frequency and attenuation rate of the k'th mode. In epicentral coordinates, A_k, ϕ_k includes both the source excitation and the geometrical mode behavior as a function of epicentral distance.


Deep earthquake at PFO
Spectral amplitude
50
40
30
20
10
0
1
2
3
4
5
6
Frequency (mHz)

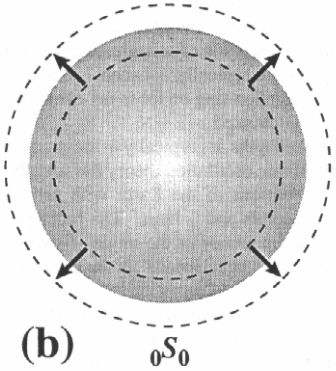

Shallow earthquake at CMO
Spectral amplitude
3.0
2.5
2.0
1.5
1.0
0.5
0.0
1
2
3
4
5
6
Frequency (mHz)


SSB 1992.347 dist: 115° azi: 76°
R1
R2
R3
R4
R5
R6
R7
R8
1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
12
R1
X1
X2
R2
1.0
0.5
0.0
−0.5
−1.0
0.0
1.0
2.0
3.0
time/h

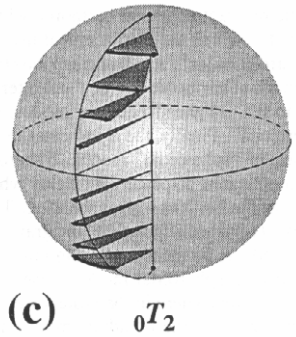
Spheroidal



Radial



Toroidal



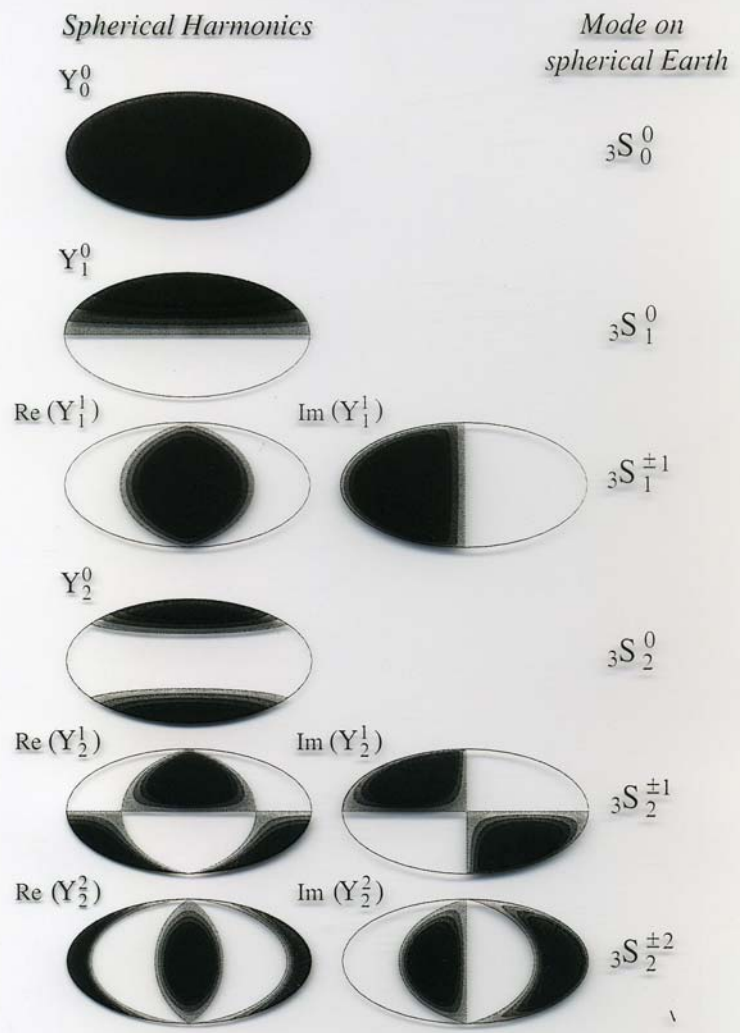
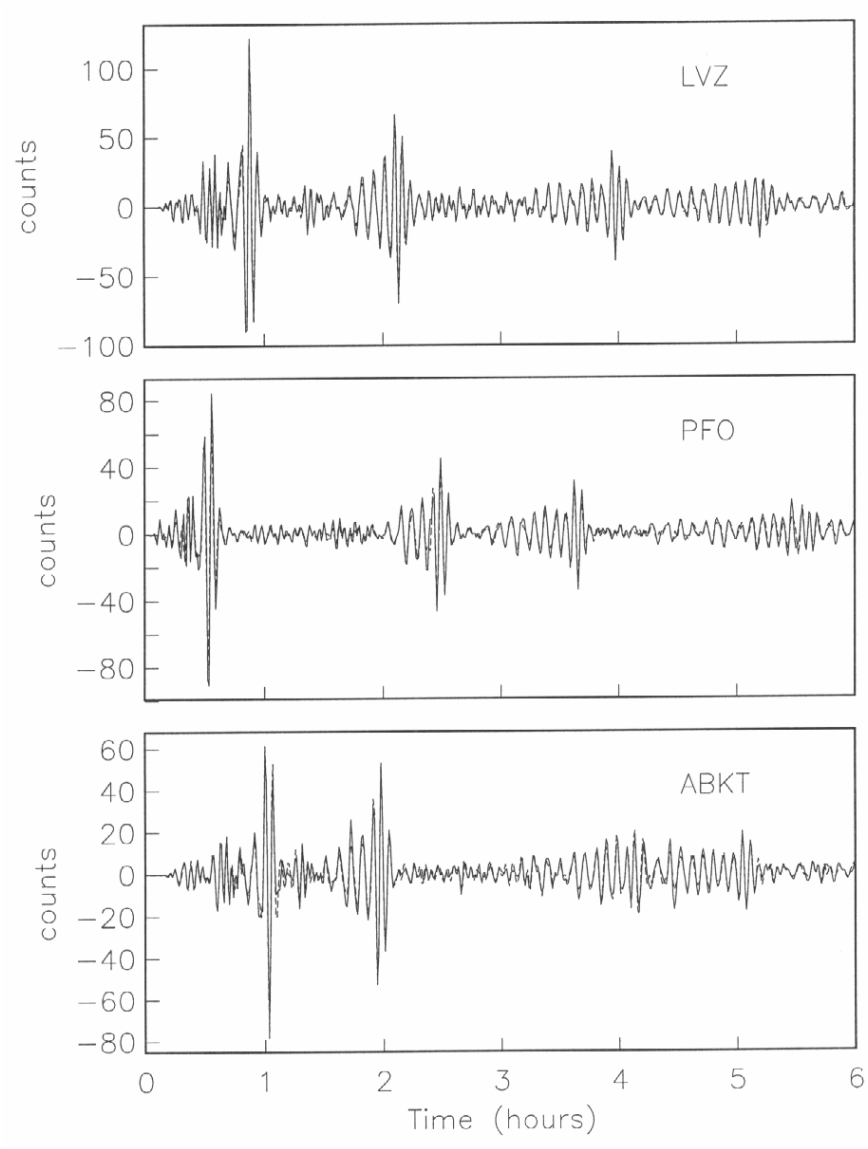


Figure 17 Some low-order spherical harmonics plotted in Hammer-Aitoff projection. Note that the singlets of a spheroidal mode have these shapes on the $\hat{\mathbf{r}}$ component of recordings on a spherical Earth (eq. (A.A3)).

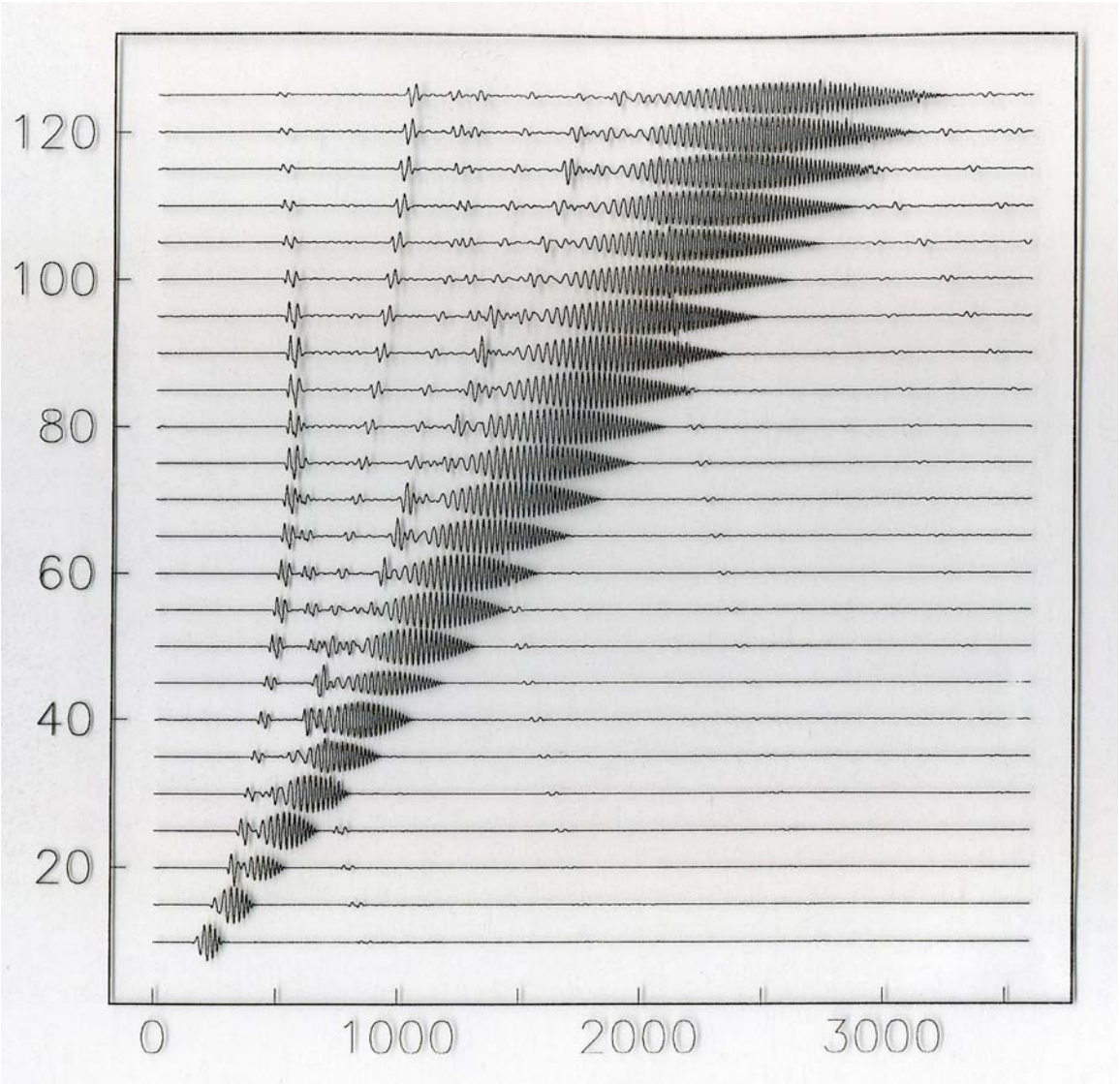
Can model real data



Bolivia, T>120 sec

Complete synthetics -- includes diffraction etc.

SH, T>5sec



Distance

Reduced time

Basic equations
(now for the fun stuff!)

Basic equations

$$\rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \nabla \cdot \mathbf{T} - \nabla(s_r \rho_0 g_0) - \rho_0 \nabla \phi_1 + \hat{\mathbf{r}} g_0 \nabla \cdot (\rho_0 \mathbf{s}) + \mathbf{f}$$
$$\nabla^2 \phi_1 = -4\pi G \nabla \cdot (\rho_0 \mathbf{s})$$

Consider case when $\mathbf{f} = 0$. Look for solutions of form:

$$\mathbf{s}(\mathbf{r}, t) = \mathbf{s}_k(\mathbf{r}) e^{i\omega_k t}; \quad \phi_1 = \phi_{1k} e^{i\omega_k t}$$

Equations to be solved are:

$$-\rho_0 \omega_k^2 \mathbf{s}_k = \nabla \cdot \mathbf{T} - \rho_0 \nabla \phi_{1k} + \hat{\mathbf{r}} g_0 \nabla \cdot (\rho_0 \mathbf{s}_k) - \nabla(s_{kr} g_0 \rho_0)$$

$$\nabla^2 \phi_{1k} = -4\pi G \nabla \cdot (\rho_0 \mathbf{s}_k)$$

We consider a SNRETI Earth model (weak anelastic behavior is included using perturbation theory.)

TI:

$$\rho_0, \mathrm{A}, \mathrm{C}, \mathrm{N}, \mathrm{L}, \mathrm{F}$$

Isotropic:

$$\rho_0, \lambda, \mu; \quad \mathrm{A} = \mathrm{C} = \lambda + 2\mu, \mathrm{N} = \mathrm{L} = \mu, \mathrm{F} = \lambda$$

Constitutive relationship

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Tranversely isotropic

$$\mathbf{C} = \begin{bmatrix} \lambda' + 2\mu' & \lambda' & \lambda & & & \\ \lambda' & \lambda' + 2\mu' & \lambda & & 0 & \\ \lambda & \lambda & \beta & & & \\ & & & \mu & & \\ & 0 & & & \mu & \\ & & & & & \mu' \end{bmatrix}$$

Isotropic

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & 0 & & & \mu & \\ & & & & & \mu \end{bmatrix}$$

Note that there are five independent elastic coefficients for the transversely isotropic case (as opposed to two for the isotropic case). A commonly used alternative notation is A $= \lambda' + 2\mu'$, C $= \beta$, F $= \lambda$, L $= \mu$, and N $= \mu'$.

Boundary conditions

- Two types of boundaries: welded (e.g. 660) and free-slip (e.g. CMB)

- Boundary conditions involve continuity of displacement and tractions on the deformed boundaries of the model. These are linearized onto the undeformed boundaries. Note that the traction vector is given by $\mathbf{t} = \hat{\mathbf{r}} \cdot \mathbf{T}$

A summary of the boundary conditions on the undeformed boundary with an isotropic prestress is

$\hat{\mathbf{r}} \cdot \mathbf{s}$ is continuous at all boundaries
$\mathbf{s}$ is regular at the origin
$\mathbf{s}$ is continuous at welded boundaries
ϕ_1 is continuous at all boundaries
ϕ_1 is zero at infinity
$\partial\phi_1/\partial r + 4\pi G\rho_0 s_r$ is continuous at all boundaries
$\mathbf{t}$ is continuous at all boundaries
$\mathbf{t}$ is zero at the free surface
$\mathbf{t}$ is regular at the origin.

Separating variables

Expand scalars in SH and vectors in VSH:

$$\phi_1 = \sum_{l,m} \Phi_{1l}^m(r) Y_l^m(\theta, \phi)$$

$$\mathbf{s}_k = \hat{\mathbf{r}}\ {}_kU + \nabla_1\ {}_kV - \hat{\mathbf{r}} \times (\nabla_1\ {}_kW)$$

$$\mathbf{t}_k = \hat{\mathbf{r}}\ {}_kR + \nabla_1\ {}_kS - \hat{\mathbf{r}} \times (\nabla_1\ {}_kT)$$

where ${}_kU$, ${}_kV$, ${}_kW$, ${}_kR$, ${}_kS$, and ${}_kT$ are scalars which are expanded in ordinary spherical harmonics, e.g.,

$${}_kU = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} {}_kU_l^m(r) Y_l^m(\theta, \phi)$$

where the $Y_l^m(\theta, \phi)$ are fully normalized spherical harmonics.

Separating variables: Poisson's equation

Consider

$$\nabla^2 \phi_{1k} = -4\pi G \nabla \cdot (\rho_0 \mathbf{s}_k)$$

Substituting in the spherical harmonic expansions and supressing the indices on ${}_kU_l^m$, etc., gives

$$\frac{1}{r^2}\left(\frac{d}{dr} r^2 \frac{d\Phi_1}{dr}\right) - l(l+1)\frac{\Phi_1}{r^2} = -4\pi G\left[(\rho_0 U)' + \rho_0 F\right]$$

where $F = (2U - l(l+1)V)/r$ and prime ($'$) indicates radial derivative. Note that we can use ordinary derivatives because U, V, W and Φ_1 are functions only of radius and the dependence on θ and ϕ has been elliminated.

Separating variables – continued

Now consider our other equation:

$$-\rho_0\omega_k^2\mathbf{s}_k = \nabla\cdot\mathbf{T} - \rho_0\nabla\phi_{1k} + \hat{\mathbf{r}}g_0\nabla\cdot(\rho_0\mathbf{s}_k) - \nabla(s_{kr}g_0\rho_0)$$

On doing the vector spherical harmonics and separating variables gives:

$$\begin{aligned}
-\rho_0\omega_k^2 U &= R' - \rho_0\Phi_1' + g_0\left((\rho_0 U)' + \rho_0 F\right) - (\rho_0 g_0 U)' \\
&\quad - \frac{1}{r}\left[2(\mathrm{F}-\mathrm{C})U' + 2(\mathrm{A}-\mathrm{N}-\mathrm{F})F + l(l+1)S\right] \\
-\rho_0\omega_k^2 V &= S' - \frac{\rho_0\Phi_1}{r} - \frac{\rho_0 g_0 U}{r} \\
&\quad + \frac{1}{r}\left[(\mathrm{A}-\mathrm{N})F + \mathrm{F}U' + 3S - \frac{\mathrm{N}V}{r}(l+2)(l-1)\right] \\
-\rho_0\omega_k^2 W &= T' + \frac{1}{r}\left[3T - \frac{\mathrm{N}W}{r}(l+2)(l-1)\right]
\end{aligned}$$

where

$$R = \mathrm{C}U' + \mathrm{F}F \quad \text{and} \quad S = \mathrm{L}(V' + \frac{U-V}{r})$$

$$\text{and} \quad T = \mathrm{L}(W' - \frac{W}{r})$$

Toroidal modes

Toroidal modes

$$\frac{dW}{dr} = \frac{T}{\mathrm{L}} + \frac{W}{r}$$

and $$\frac{dT}{dr} = \frac{\mathrm{N}}{r^2}(l+2)(l-1)W - \rho_0\omega_k^2 W - \frac{3T}{r}$$

which can be written in matrix form as

$$\frac{d}{dr}\begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} 1/r & 1/L \\ \mathrm{N}(l+2)(l-1)/r^2 - \rho_0\omega_k^2 & -3/r \end{bmatrix}\begin{bmatrix} W \\ T \end{bmatrix}$$

This form is convenient for numerical solution.

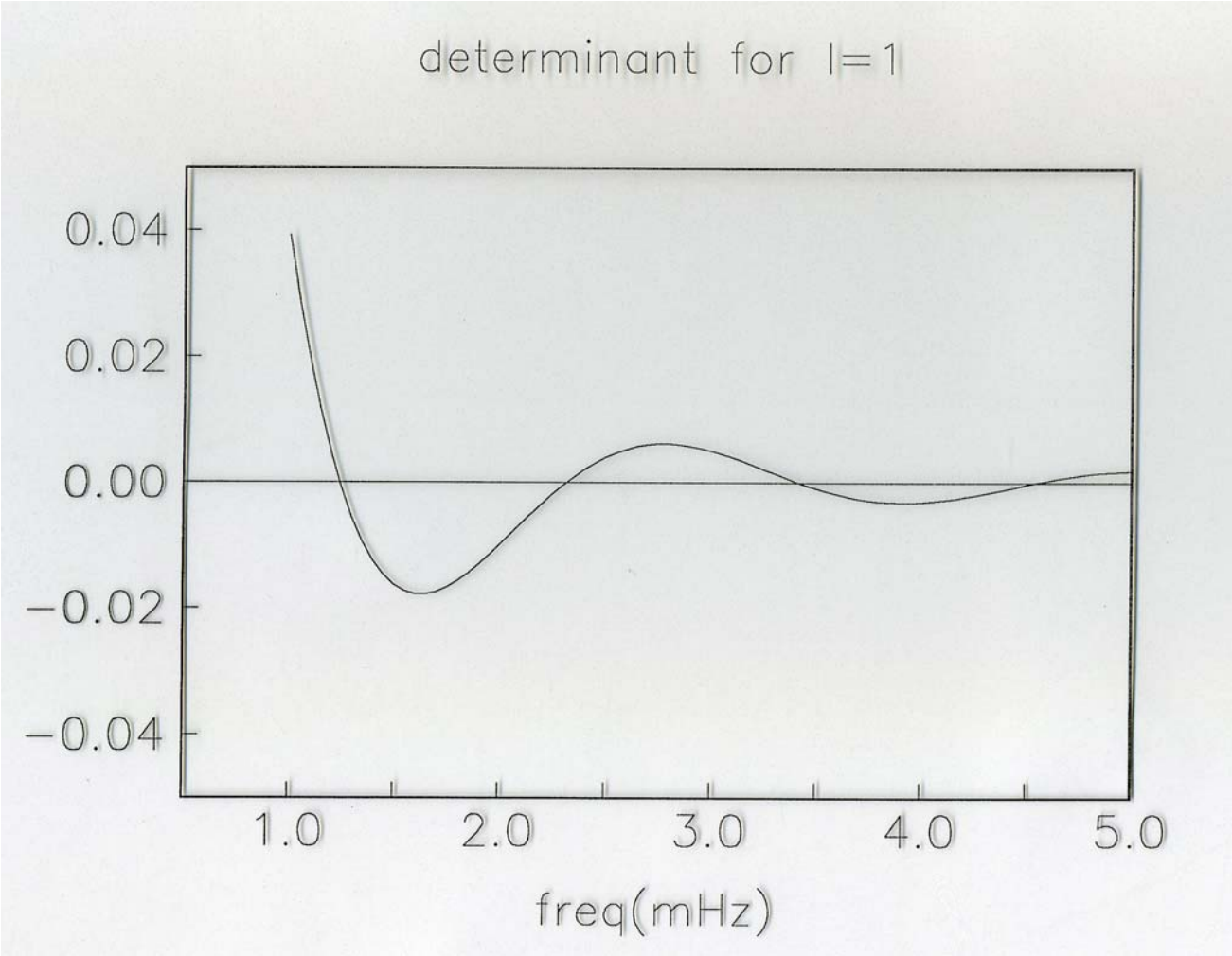
(W is scalar for displacement, T is scalar for traction)

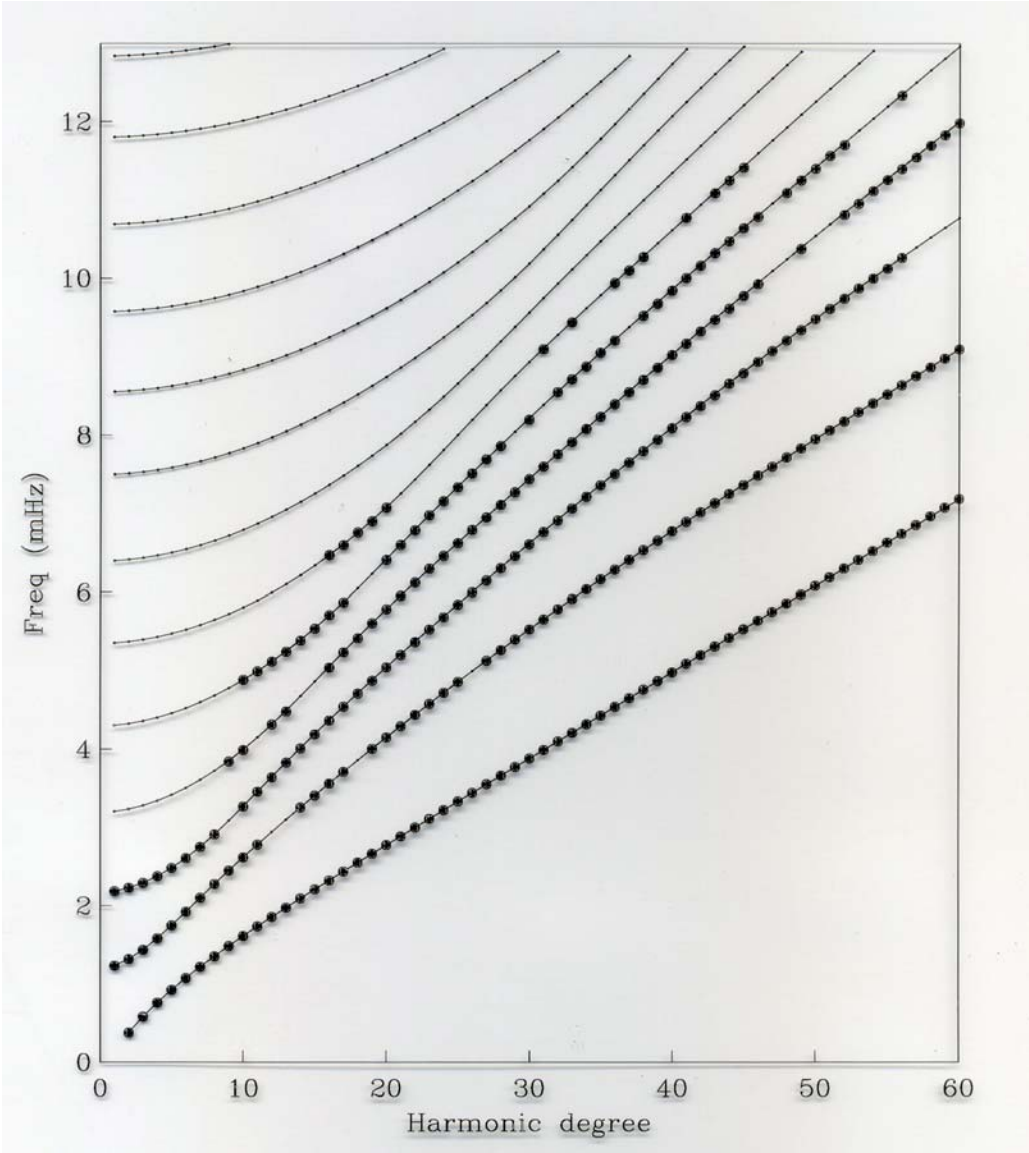
Note that matrix does not depend on m

Algorithm for toroidal modes

- Choose harmonic degree and frequency
- Compute starting solution for (W,T)
- Integrate equations to top of solid region
- Is T(surface)=0? No: go change frequency and start again. Yes: we have a mode solution

T(surface) for harmonic degree 1





(black dots are observed modes)

Toroidal modes of a homogeneous sphere or shell

Consider isotropic case L = N = μ. For μ constant:

$$\frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} + W\left(\frac{\omega_k^2}{V_s^2} - \frac{l(l+1)}{r^2}\right) = 0$$

This is the equation for spherical Bessel functions with solutions: $j_l(kr)$ and $y_l(kr)$ where $k = \omega/V_s$. In general, W is a linear combination of j_l and y_l, i.e.,

$$W(r) = Aj_l(kr) + By_l(kr)$$

If we consider a homogeneous *sphere*, $B = 0$ because the solutions must be regular at the origin.

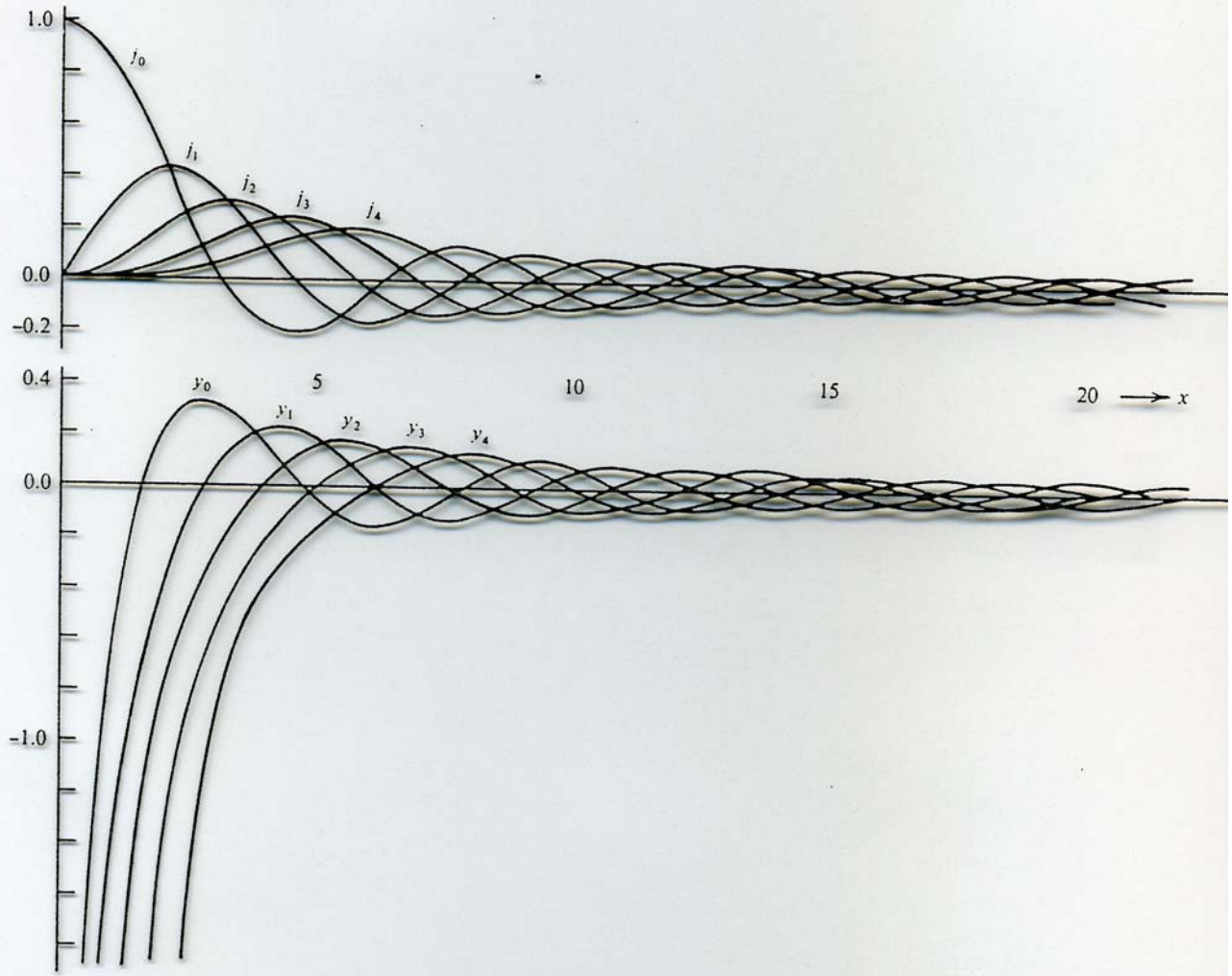


Fig. 2.1. Spherical Bessel functions j_l and y_l for $l = 0, 1, 2, 3, 4$.

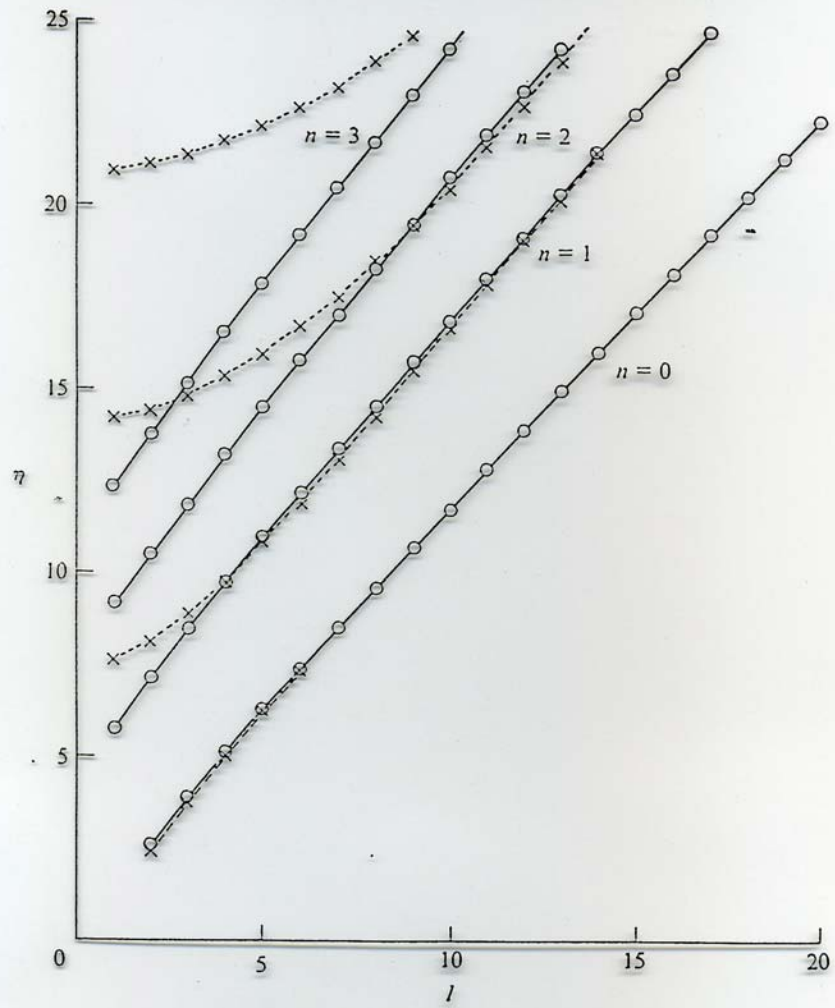


Fig. Non-dimensional frequency $\eta(= ka = \omega a/c_s)$ of toroidal oscillations as function of l. Open circles and solid curves refer to a uniform sphere and crosses and dashed curves to a uniform shell.

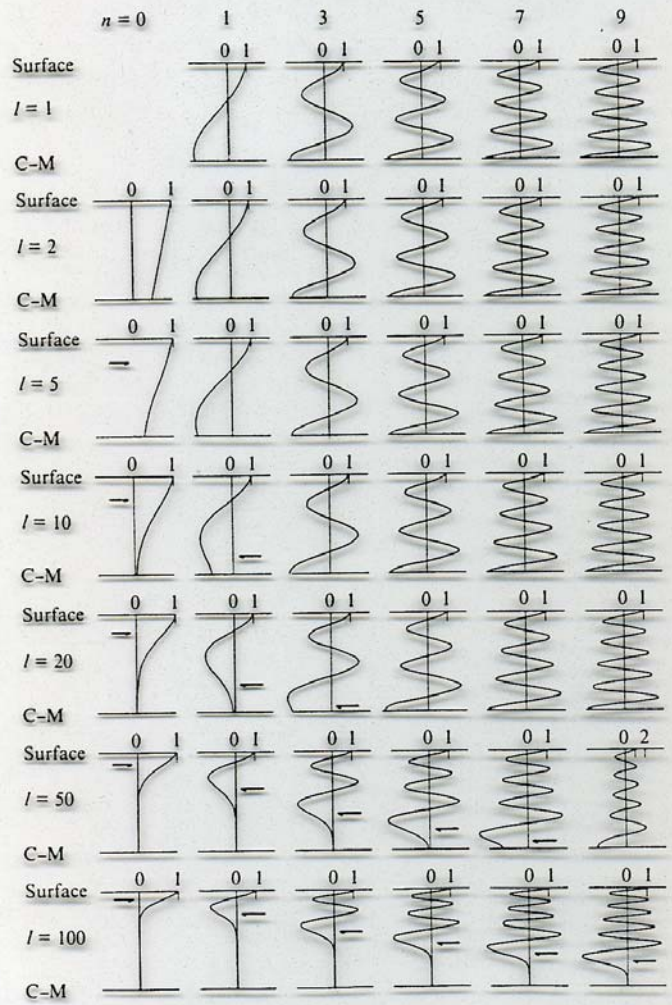


Fig. 3.3. Radial distribution of amplitude of eigenfunctions of toroidal oscillations for a uniform shell. Surface denotes the outer surface and C–M the inner surface (core–mantle boundary) of a uniform shell. The amplitude at the surface is taken as unity.

Some mode properties

Consider toroidal modes (Sturm-Liouville). Let $v_k = W/r$ then

$$(\mu r^4 y_k')' + r^2 \left[\rho_0 \omega_k^2 r^2 - \mu(l+2)(l-1)\right] y_k = 0$$

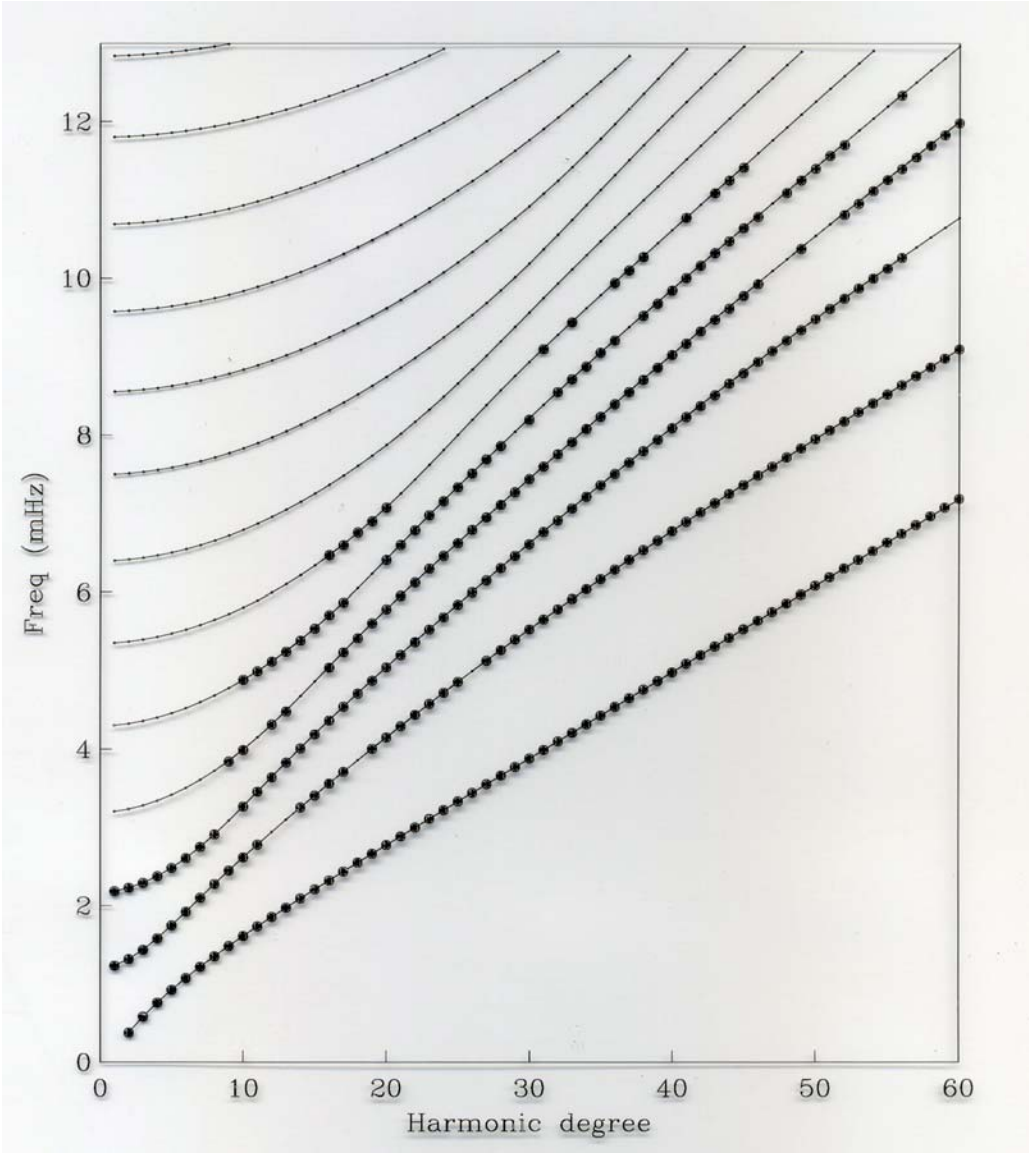
If term in brackets is positive/negative, y_k is oscillatory/exponential. The "turning point" is when

$$\rho_0 \omega_k^2 r^2 - \mu(l+2)(l-1) = 0$$

Asymptotically for large l, $ka = \omega p = l + \frac{1}{2}$ where k is wavenumber and $p = 1/c$ is phase slowness. Then, because $(l+2)(l-1) \simeq (l + \frac{1}{2})^2$ and $\mu = \rho_0 V_s^2$, at turning point

$$\rho_0 \omega_k^2 r^2 - \mu \omega_k^2 p^2 = 0 \rightarrow p = \frac{r}{V_s}$$

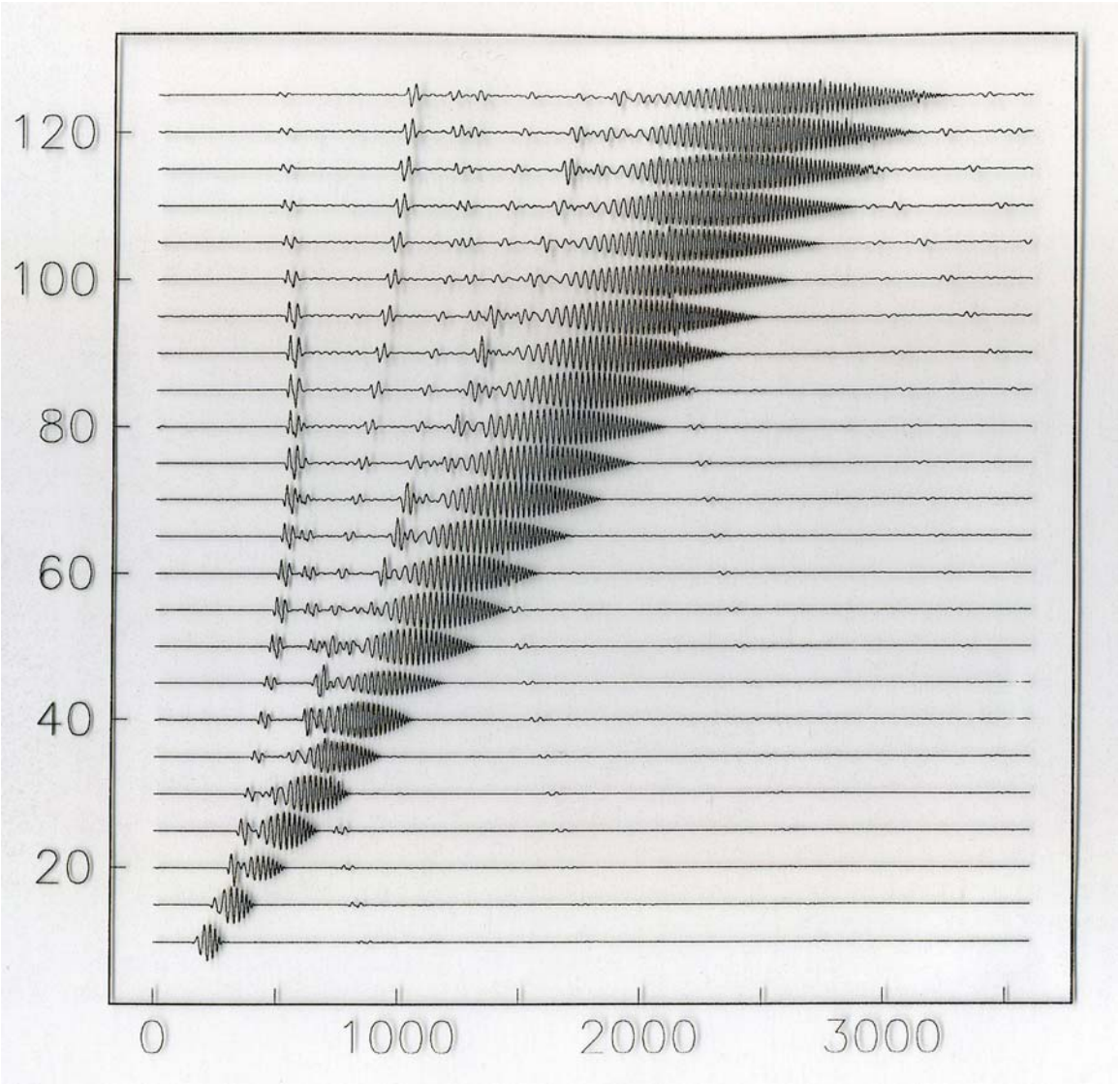
p is also equivalent to the ray parameter.
Lines of constant ray parameter are straight lines on an ω/l diagram



(black dots are observed modes)

Complete synthetics -- includes diffraction etc.

SH, T>5sec



Distance

Reduced time

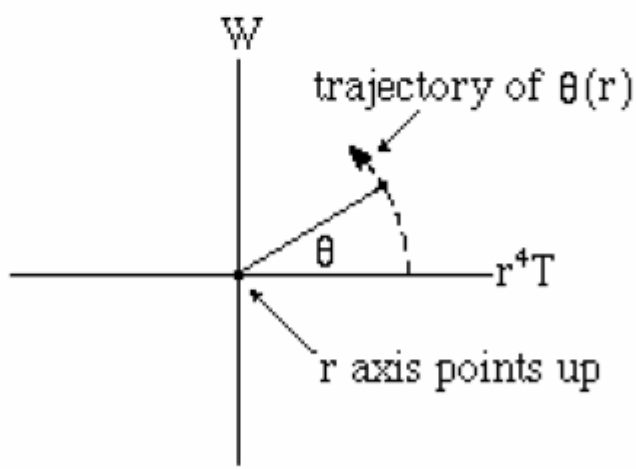
Mode counting

For toroidal modes, make Prüfer substitution

$$\tan\theta = \frac{W}{r^4 T}$$

$$\frac{d\theta}{dr} = r^2\left[\omega_k^2\rho_0 r^2 - \mu(l+2)(l-1)\right]\sin^2\theta + \frac{1}{\mu r^4}\cos^2\theta$$

Above turning point, RHS is always positive so $\theta(r)$ always increases and the zeroes of W and T must interlace



For spheroidal modes, we get interlacing behavior of combinations of minor elements.

Radial and Spheroidal modes

Equations for spheroidal modes

Three coupled second order ODEs (3 solutions regular at the origin):

$$
\begin{aligned}
-\rho_0\omega_k^2 U &= R' - \rho_0\Phi_1' + g_0\left((\rho_0 U)' + \rho_0 F\right) - (\rho_0 g_0 U)' \\
&\quad - \frac{1}{r}\left[2(\mathrm{F}-\mathrm{C})U' + 2(\mathrm{A}-\mathrm{N}-\mathrm{F})F + l(l+1)S\right] \\
-\rho_0\omega_k^2 V &= S' - \frac{\rho_0\Phi_1}{r} - \frac{\rho_0 g_0 U}{r} \\
&\quad + \frac{1}{r}\left[(\mathrm{A}-\mathrm{N})F + \mathrm{F}U' + 3S - \frac{\mathrm{N}V}{r}(l+2)(l-1)\right]
\end{aligned}
$$

$$
\frac{1}{r^2}\left(\frac{d}{dr}r^2\frac{d\Phi_1}{dr}\right) - l(l+1)\frac{\Phi_1}{r^2} = -4\pi G\left[(\rho_0 U)' + \rho_0 F\right]
$$

where

$$
R = \mathrm{C}U' + \mathrm{F}F \quad \text{and} \quad S = \mathrm{L}(V' + \frac{U-V}{r})
$$

Spheroidal modes

Convenient to define:

$$\Psi_1 = \frac{d\Phi_1}{dr} + \frac{(l+1)}{r}\Phi_1 + 4\pi G\rho_0 U$$

which is continuous and zero at the free surface. Then

$$\frac{d\mathbf{y}}{dr} = \mathbf{A}\,\mathbf{y}$$

The 6×6 coefficient matrix $\mathbf{A}$ takes on a highly symmetric form if we are careful in our definition of $\mathbf{y}$. We choose as our vector

$$\mathbf{y} = \begin{bmatrix} rU \\ rV\mathcal{L} \\ r\Phi_1 \\ rR \\ rS\mathcal{L} \\ r\Psi_1/4\pi G \end{bmatrix}$$

Note that y_4, y_5, and y_6 at the free surface. The matrix, $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} -T & C \\ S & T^T \end{bmatrix}$$

Only 14 distinct non-zero elements of A

$$T_{11} = \frac{2\mathrm{F}}{\mathrm{C}r} - \frac{1}{r} \quad T_{12} = -\mathcal{L}\frac{\mathrm{F}}{\mathrm{C}r}$$

$$T_{21} = \frac{\mathcal{L}}{r} \quad T_{22} = -\frac{2}{r}$$

$$T_{31} = 4\pi G\rho_0 \quad T_{33} = \frac{l}{r}$$

$$C_{11} = \frac{1}{\mathrm{C}} \quad C_{22} = \frac{1}{\mathrm{L}} \quad C_{33} = 4\pi G$$

$$S_{11} = -\rho_0\omega_k^2 + \frac{4}{r^2}(\gamma - rg_0\rho_0)$$

$$S_{22} = -\rho_0\omega_k^2 + \frac{\mathcal{L}^2(\gamma + \mathrm{N}) - 2\mathrm{N}}{r^2}$$

$$S_{12} = S_{21} = \frac{\mathcal{L}}{r^2}(rg_0\rho_0 - 2\gamma)$$

$$S_{13} = S_{31} = -\frac{\rho_0(l+1)}{r} \quad S_{23} = S_{32} = \frac{\mathcal{L}\rho_0}{r}$$

where $\mathcal{L} = \sqrt{l(l+1)}$ and $\gamma = \mathrm{A} - \mathrm{N} - \mathrm{F}^2/\mathrm{C}$.

Fluid (isotropic) regions with self-gravitation

Now have N = L = $\mu = 0; y_5 = 0$. We end up with a system of four equations to solve. Choose

$$\mathbf{y} = \begin{bmatrix} rU \\ r\Phi_1 \\ rR \\ r\Psi_1/4\pi G \end{bmatrix}$$

Then

$$\frac{d\mathbf{y}}{dr} = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} -T & C \\ S & T^T \end{bmatrix} \mathbf{y}$$

with

$$T_{11} = -g_0\gamma + \frac{1}{r} \quad T_{12} = -\gamma \quad T_{21} = 4\pi G\rho_0 \quad T_{22} = \frac{l}{r}$$

$$C_{11} = \frac{1}{\lambda} - \frac{\gamma}{\rho_0} \quad C_{22} = 4\pi G$$

$$S_{11} = -\rho_0 \left[\omega_k^2 + \frac{4g_0}{r} - g_0^2\gamma\right] \quad S_{22} = \rho_0\gamma$$

$$S_{12} = S_{21} = \rho_0 \left[g_0\gamma - \frac{(l+1)}{r}\right]$$

where $\mathcal{L} = \sqrt{l(l+1)}$ and $\gamma = \mathcal{L}^2/(r^2\omega_k^2)$

Radial modes ($l = 0$)

V and S are zero so equations simplify:

$$\frac{dy_1}{dr} = -\left(\frac{2\mathrm{F}}{\mathrm{C}r} - \frac{1}{r}\right) y_1 + \frac{y_4}{\mathrm{C}}$$

$$\frac{dy_4}{dr} = \left[-\rho_0\omega_k^2 + \frac{4}{r^2}(\gamma - r g_0 \rho_0)\right] y_1 + \left(\frac{2\mathrm{F}}{\mathrm{C}r} - \frac{1}{r}\right) y_4$$

$$\text{with} \quad \frac{dy_6}{dr} = -\frac{\rho_0}{r} y_1 \quad \text{and} \quad y_3 = 4\pi G r y_6$$

where $y_1 = rU, y_4 = rR, y_3 = r\Phi_1,$ and $y_6 = r\Psi_1/4\pi G.$

(Solution follows that of toroidal modes)

Radial modes: Homogeneous isotropic sphere

Equation reduces to

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} + \left[k^2 - \frac{2}{r^2}\right]U = 0$$

where

$$k^2 = \rho_0 \frac{\frac{16}{3}\pi G\rho_0 + \omega_k^2}{\lambda + 2\mu}$$

This is the differential equation satisfied by spherical Bessel functions of the first order. Because the solution is regular at the origin we may take

$$U = A\left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right) = Aj_1(kr)$$

where A is an arbitrary constant. This may be readily differentiated to give R.

(gravity term corresponds to a frequency of about 0.4mHz)

Spheroidal modes

Spheroidal modes

- Numerical problems: illustrate with homogeneous sphere
- Choose l and make guess of ω_k
- There are 3 solutions regular at the origin. We integrate these to the free surface of the sphere
- If we have chosen ω_k correctly then R, S, and Ψ_1 should all be zero at the free surface or, equivalently, y_4, y_5, and y_6 will all be zero. Thus we require

$$\mathbf{y} = a_1\mathbf{y}^1 + a_2\mathbf{y}^2 + a_3\mathbf{y}^3$$

where the a's are arbitrary constants such that

$$a_1 \begin{bmatrix} y_4^1 \\ y_5^1 \\ y_6^1 \end{bmatrix} + a_2 \begin{bmatrix} y_4^2 \\ y_5^2 \\ y_6^2 \end{bmatrix} + a_3 \begin{bmatrix} y_4^3 \\ y_5^3 \\ y_6^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or equivalently

$$\det \begin{bmatrix} y_4^1 & y_4^2 & y_4^3 \\ y_5^1 & y_5^2 & y_5^3 \\ y_6^1 & y_6^2 & y_6^3 \end{bmatrix} = 0$$

If this is not true we must start again with a new value of ω_k.

- Computing the determinant is a numerical disaster

Minors (at last)

- To simplify matters, we will consider the spheroidal mode equations in the Cowling approximation where we include all buoyancy terms but ignore perturbations to the gravitational potential

Spheroidal modes – no self graviation

Set $d\Phi_1/dr$ and Φ_1 to zero. ($\Psi_1 = 4\pi G\rho_0 U$), Choose

$$\mathbf{y} = \begin{bmatrix} rU \\ rV\mathcal{L} \\ rR \\ rS\mathcal{L} \end{bmatrix}$$

Then

$$\frac{d\mathbf{y}}{dr} = \mathbf{A}\,\mathbf{y} = \begin{bmatrix} -T & C \\ S & T^T \end{bmatrix} \mathbf{y}$$

with

$$T_{11} = \frac{2\mathrm{F}}{\mathrm{C}r} - \frac{1}{r} \quad T_{12} = -\mathcal{L}\frac{\mathrm{F}}{\mathrm{C}r}$$

$$T_{21} = \frac{\mathcal{L}}{r} \quad T_{22} = -\frac{2}{r}$$

$$C_{11} = \frac{1}{\mathrm{C}} \quad C_{22} = \frac{1}{\mathrm{L}}$$

$$S_{11} = -\rho_0\omega_k^2 + \frac{4}{r^2}(\gamma - rg_0\rho_0) + 4\pi G\rho_0$$

$$S_{22} = -\rho_0\omega_k^2 + \frac{\mathcal{L}^2(\gamma + \mathrm{N}) - 2\mathrm{N}}{r^2}$$

$$S_{12} = S_{21} = \frac{\mathcal{L}}{r^2}(rg_0\rho_0 - 2\gamma)$$

where $\mathcal{L} = \sqrt{l(l+1)}$ and $\gamma = \mathrm{A} - \mathrm{N} - \mathrm{F}^2/\mathrm{C}$.

Why minors?

Consider solid sphere with no self-gravitation (system of 4 equations). Two solutions $\mathbf{y}^1$, $\mathbf{y}^2$ satisfy regularity at the origin. Final solution $\mathbf{y}$ is a combination of these such that y_3, y_4 are zero at the free surface at a root:

$$\begin{aligned} y_3 =& a_1 y_3^1 + a_2 y_3^2 = 0 \\ y_4 =& a_1 y_4^1 + a_2 y_4^2 = 0 \end{aligned}$$

or

$$\det \begin{bmatrix} y_3^1 & y_3^2 \\ y_4^1 & y_4^2 \end{bmatrix} = y_3^1 y_4^2 - y_4^1 y_3^2 = 0$$

This last step, the formation of the determinant, is the one that causes us numerical problems – we are often differencing two large, almost identical numbers with disastrous consequences.

Minors – continued

We work directly with a vector of second-order minors, *i.e.*, let the minor vector of $[\mathbf{y}^1, \mathbf{y}^2]$ be

$$\begin{bmatrix} y_1^1, y_1^2 \\ y_2^1, y_2^2 \\ y_3^1, y_3^2 \\ y_4^1, y_4^2 \end{bmatrix} = \begin{bmatrix} y_1^1 y_2^2 - y_2^1 y_1^2 \\ y_1^1 y_3^2 - y_3^1 y_1^2 \\ y_1^1 y_4^2 - y_4^1 y_1^2 \\ y_2^1 y_3^2 - y_3^1 y_2^2 \\ y_2^1 y_4^2 - y_4^1 y_2^2 \\ y_3^1 y_4^2 - y_4^1 y_3^2 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}$$

$\mathbf{m}$ is the vector of all independent second-order minors of $\mathbf{y}^1$ with $\mathbf{y}^2$. It follows that $d\mathbf{m}/dr = \mathbf{B}\mathbf{m}$, where $\mathbf{B}$ is

$$\begin{bmatrix} -T^+ & 0 & C_{22} & -C_{11} & 0 & 0 \\ S_{12} & 0 & T_{21} & -T_{12} & 0 & 0 \\ S_{22} & T_{12} & T^- & 0 & -T_{12} & C_{11} \\ -S_{11} & -T_{21} & 0 & -T^- & T_{21} & -C_{22} \\ -S_{12} & 0 & -T_{21} & T_{12} & 0 & 0 \\ 0 & -S_{12} & S_{11} & -S_{22} & S_{12} & T^+ \end{bmatrix}$$

where $T^+ = T_{22} + T_{11}$ and $T^- = T_{22} - T_{11}$

Note $m_2 = -m_5$ everywhere (so we only need propagate a 5-vector) and $m_6 = 0$ at the free surface at a root.

Mode counting with minors

- Need an interlacing theorem – use the following identity which the minor vector satisfies everywhere:

$$m_1 m_6 - m_2 m_5 + m_3 m_4 \equiv m_1 m_6 + m_2^2 + m_3 m_4 = 0$$

- If $m_3 = m_4$ and $m_6 = 0$ then the whole minor vector disappears (maybe). This can't happen which implies that the zeroes of m_6 and $m_3 - m_4$ must interlace (just like for W and T for toroidal modes).

- A mode counter based on this means identifying the zeroes of m_6 as we integrate the equations through the Earth. This can be numerically troublesome since the minor vector does not oscillate about zero and the zeroes of m_6 can be arbitrarily close in radius. In this case, the mode count is off by 2. The code attempts to fix itself if this happens.

Minors – computation of eigenfunction

The eigenfunction, $\mathbf{y}$, is a linear combination of the two solutions $\mathbf{y}^1$ and $\mathbf{y}^2$, *i.e.*,

$$\mathbf{y} = a_1\mathbf{y}^1 + a_2\mathbf{y}^2$$

a_1 and a_2 can be eliminated in four different ways from this equation and we write the result in matrix form as

$$\mathbf{N} \cdot \mathbf{y} = 0$$

where

$$\mathbf{N} = \begin{bmatrix} -m_2 & -m_3 & 0 & m_1 \\ -m_4 & m_2 & -m_1 & 0 \\ 0 & -m_6 & -m_2 & -m_4 \\ m_6 & 0 & -m_3 & m_2 \end{bmatrix}$$

We integrate a solution of the original system downwards with an arbitrary starting value. Label this solution as $\mathbf{x}$ and let $\mathbf{x}(a) = (1, 0, 0, 0)$ say. At every depth form

$$\mathbf{y} = \mathbf{N} \cdot \mathbf{x} \quad \text{where} \quad \frac{d\mathbf{x}}{dr} = \mathbf{A}\mathbf{x}$$

By construction, $\mathbf{y}$ will satisfy all the boundary conditions and $\mathbf{N} \cdot \mathbf{y} = 0$ because $\mathbf{N} \cdot \mathbf{N} = 0$. $\mathbf{y}$ is therefore our desired eigenvector. Stable for "usual" modes.

Minors – with self gravitation

The calculation of minor vectors for the solid system is much more complicated. The $\mathbf{m}$ vector is now 20 elements long, *i.e.*,

$$
\begin{aligned}
& y_1^1 y_2^2 y_3^3 + y_2^1 y_3^2 y_1^3 + y_3^1 y_1^2 y_2^3 - y_1^1 y_3^2 y_2^3 - y_2^1 y_1^2 y_3^3 - y_3^1 y_2^2 y_1^3 \\
& \vdots \\
& y_4^1 y_5^2 y_6^3 + y_5^1 y_6^2 y_4^3 + y_6^1 y_4^2 y_5^3 - y_4^1 y_6^2 y_5^3 - y_5^1 y_4^2 y_6^3 - y_6^1 y_5^2 y_4^3
\end{aligned}
$$

This vector is all possible independent third-order minors of the three solutions, *i.e.*,

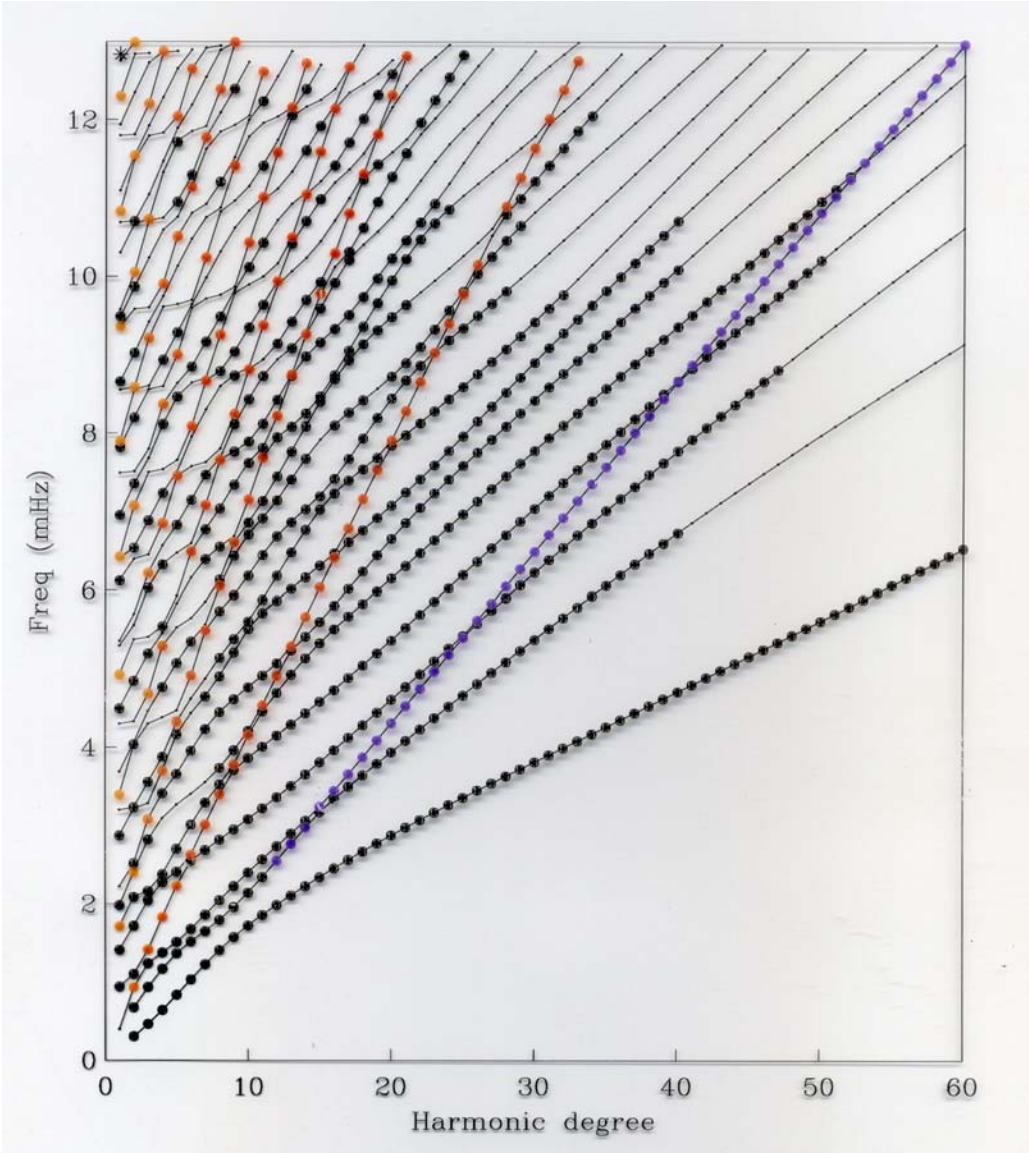
$$
\begin{array}{c}
Row1 \\ Row2 \\ Row3 \\ Row4 \\ Row5 \\ Row6
\end{array}
\begin{bmatrix}
y_1^1 & y_1^2 & y_1^3 \\
y_1^1 & y_2^2 & y_2^3 \\
y_1^1 & y_3^2 & y_3^3 \\
y_1^1 & y_4^2 & y_4^3 \\
y_1^1 & y_5^2 & y_5^3 \\
y_1^1 & y_6^2 & y_6^3
\end{bmatrix}
$$

Now compute third-order determinants using the following row combinations: 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456. Note that m_{20} is the minor vector element that should be zero at the free surface if we are at a root. There are only 14 independent minors because $m_2 = m_{13}, m_3 = -m_7, m_5 = -m_{12}, m_8 = m_{19}, m_9 = -m_{16}$ and $m_{14} = -m_{18}$.

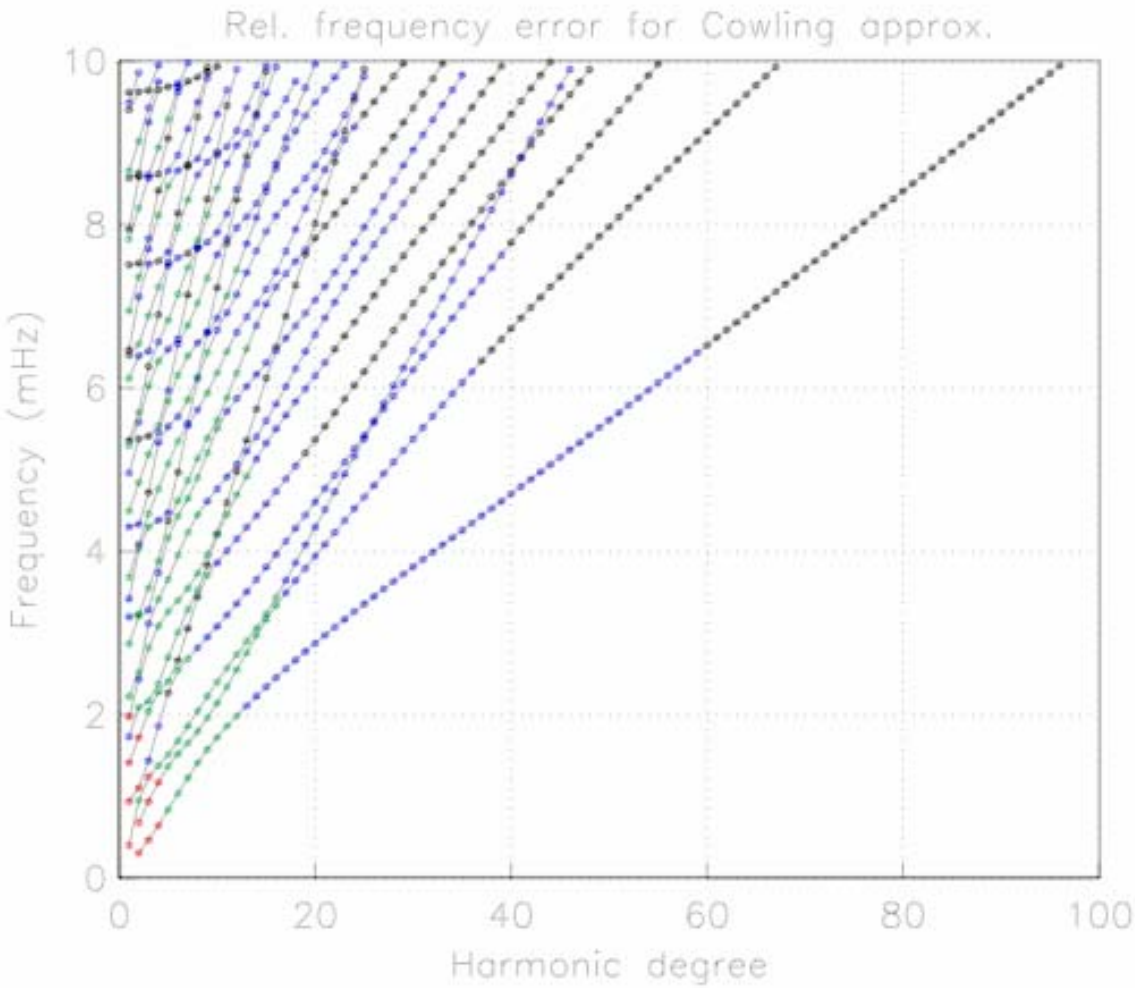
Spheroidal modes w/ self grav

$$
\begin{aligned}
m_1' =& (T_{11} + T_{22} + T_{33})m_1 + C_{33}m_4 - C_{22}m_6 + C_{11}m_{11} \\
m_2' =& S_{13}m_1 + T_{22}m_2 - T_{21}m_3 - T_{31}m_4 - C_{22}m_8 \\
m_3' =& S_{23}m_1 - T_{12}m_2 + T_{11}m_3 + C_{11}m_{18} \\
m_4' =& (T_{11} + T_{22} - T_{33})m_4 + C_{22}m_{10} - C_{11}m_{15} \\
m_5' =& -S_{12}m_1 + T_{31}m_3 + T_{33}m_5 - T_{21}m_6 - C_{33}m_9 + T_{12}m_{11} \\
m_6' =& -S_{22}m_1 - 2T_{12}m_5 + (T_{11} - T_{22} + T_{33})m_6 - C_{33}m_{10} - C_{11}m_{17} \\
m_8' =& -S_{22}m_2 + S_{12}m_3 - S_{23}m_5 + S_{13}m_6 - T_{22}m_8 + T_{31}m_{10} - T_{12}m_{18} \\
m_9' =& -S_{23}m_2 - S_{13}m_3 + S_{12}m_4 - T_{33}m_9 - T_{21}m_{10} + T_{12}m_{15} \\
m_{10}' =& -2S_{23}m_3 + S_{22}m_4 - 2T_{12}m_9 + (T_{11} - T_{22} - T_{33})m_{10} + C_{11}m_{20} \\
m_{11}' =& S_{11}m_1 - 2T_{31}m_2 + 2T_{21}m_5 + (T_{33} + T_{22} - T_{11})m_{11} - C_{33}m_{15} + C_{22}m_{17} \\
m_{15}' =& 2S_{13}m_2 - S_{11}m_4 + 2T_{21}m_9 + (T_{22} - T_{11} - T_{33})m_{15} - C_{22}m_{20} \\
m_{17}' =& 2S_{12}m_5 - S_{11}m_6 + 2T_{31}m_8 + S_{22}m_{11} + (T_{33} - T_{11} - T_{22})m_{17} + C_{33}m_{20} \\
m_{18}' =& -S_{12}m_2 + S_{11}m_3 + S_{13}m_5 - T_{21}m_8 + T_{31}m_9 + S_{23}m_{11} - T_{11}m_{18} \\
m_{20}' =& 2S_{13}m_8 - 2S_{12}m_9 + S_{11}m_{10} - S_{22}m_{15} - 2S_{23}m_{18} - (T_{11} + T_{22} + T_{33})m_{20}
\end{aligned}
$$

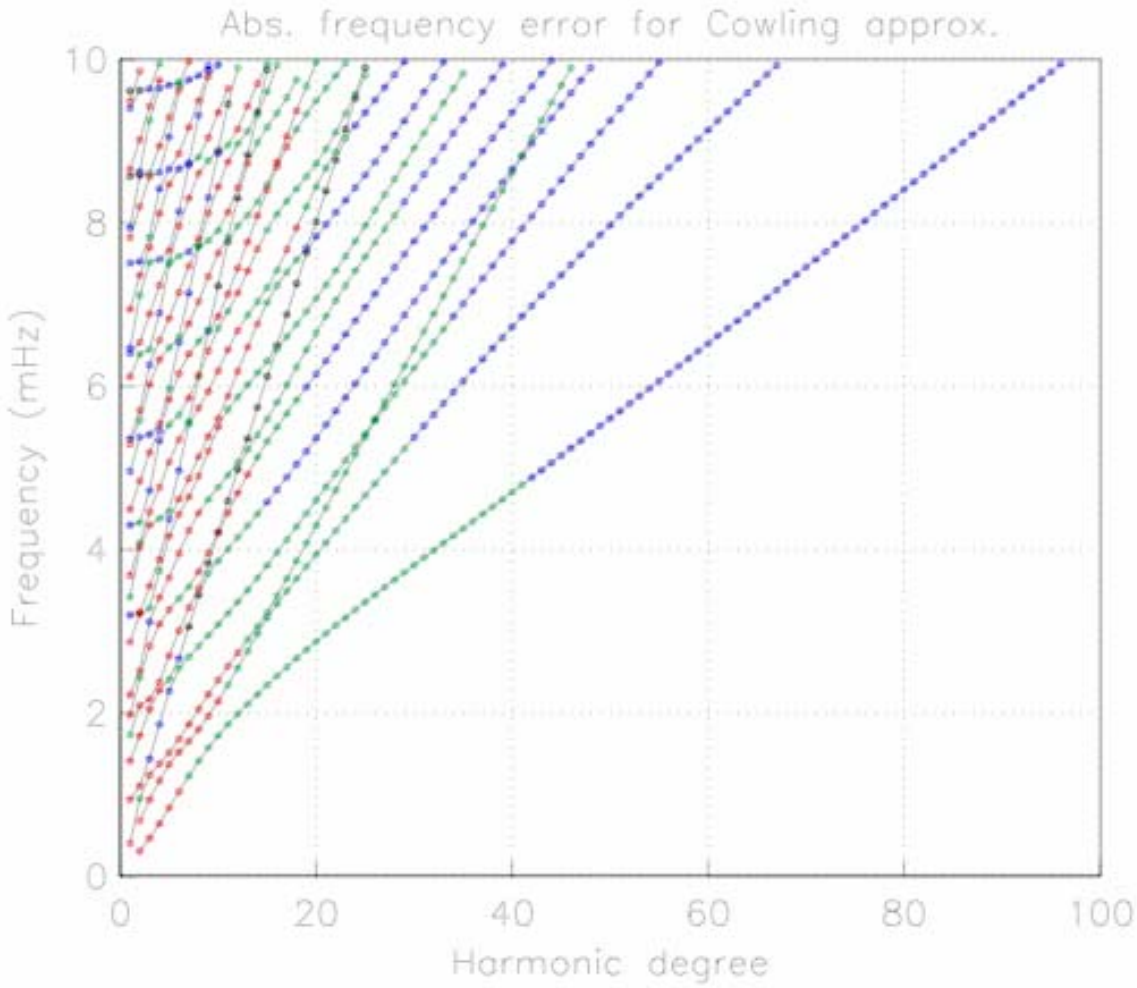
(three times slower than for Cowling approx)



(black dots are observed modes)



Red > 1%; green .1--1%; blue .01--.1%



Red>5; green 1--5; blue .1--1 microHz

Mode energy densities

Variational principles

Write basic equations as

$$\rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \mathbf{L}(\mathbf{s}) + \mathbf{f}$$

For a solution of the form $\mathbf{s} = \mathbf{s}_k(\mathbf{r}) e^{i\omega_k t}$ when $\mathbf{f} = 0$, we have

$$\mathbf{L}(\mathbf{s}_k) + \rho_0 \omega_k^2 \mathbf{s}_k = 0$$

Thus

$$\omega_k^2 \int_V \rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k \, dV = - \int_V \mathbf{s}_k^\star \cdot \mathbf{L}(\mathbf{s}_k) \, dV$$

The term on the left-hand side clearly has something to do with kinetic energy. The kinetic energy averaged over a cycle is

$$\overline{KE} = \tfrac{1}{4} \, \omega_k^2 \int_V \rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k \, dV = \omega_k^2 \mathcal{T} \quad \text{say}$$

In a similar way, the mean potential energy is

$$\mathcal{V} = -\tfrac{1}{4} \int_V \mathbf{s}^\star \cdot \mathbf{L}(\mathbf{s}) \, dV$$

There is equality of the time-averaged kinetic and potential energies. This is used as a numerical check of the quality of the calculations

Elastic Potential Energy

$$\mathcal{V} = \frac{1}{4} \int_V \{\nabla \mathbf{s}^* \cdot \cdot \mathbf{C} : \nabla \mathbf{s} + \frac{1}{2}\, \rho_0 \nabla \phi \cdot [\mathbf{s}^* \cdot \nabla \mathbf{s} - \mathbf{s}^* (\nabla \cdot \mathbf{s}) + \mathbf{s} \cdot \nabla \mathbf{s}^* - \mathbf{s}(\nabla \cdot \mathbf{s}^*)]$$

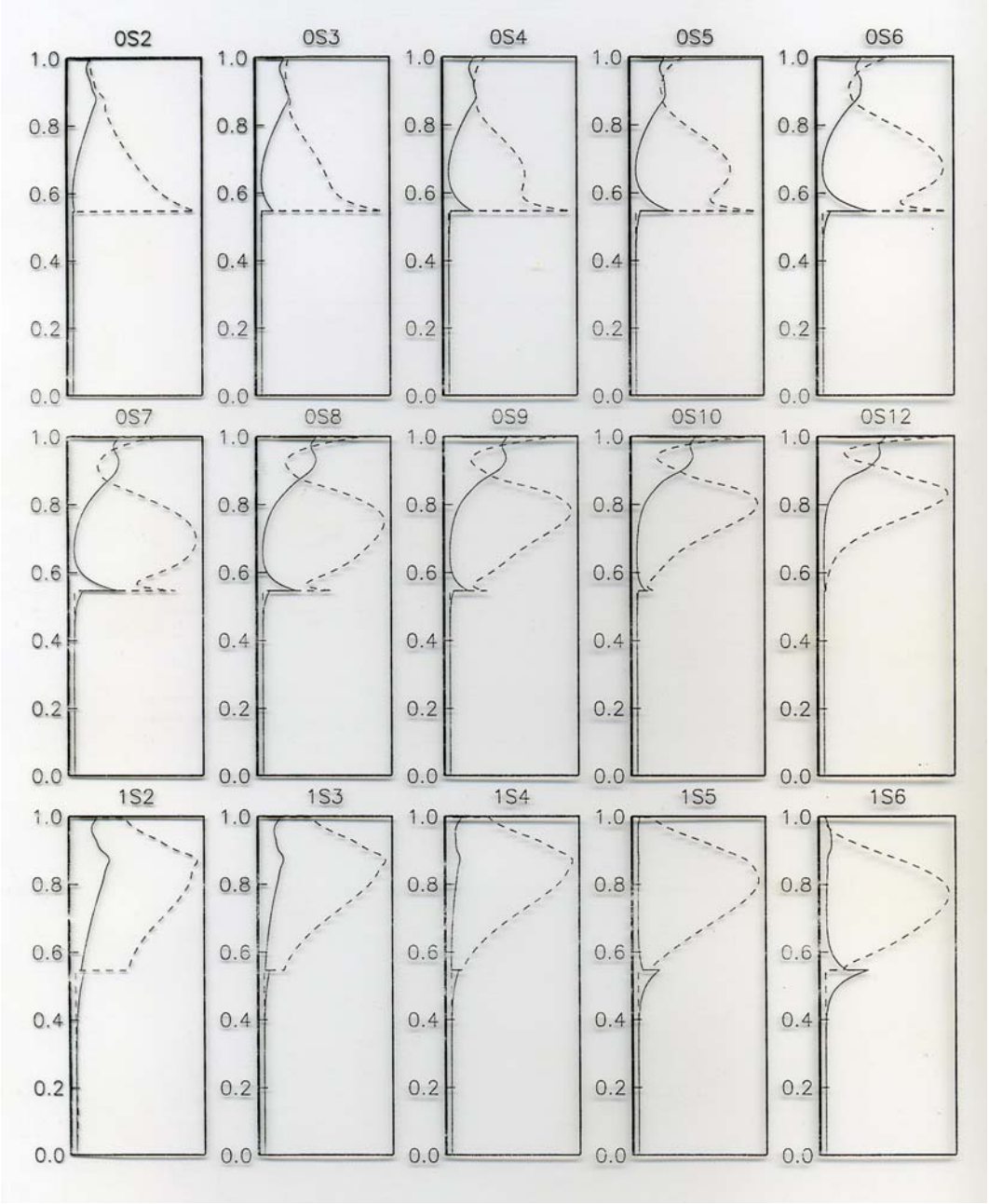
$$+ \rho_0 [\mathbf{s}^* \nabla \phi_1 + \mathbf{s} \nabla \phi_1^* + \mathbf{s}^* \cdot \mathbf{s} \nabla (\nabla \phi_1)] + \frac{1}{4\pi G} \nabla \phi_1 \cdot \nabla \phi_1^* \,\} \, dV$$

$$\begin{aligned} \int_V \boldsymbol{\epsilon}^* \cdot \cdot \mathbf{C} : \boldsymbol{\epsilon} \, dV &= \int \left[\frac{\mu}{r^2} l(l+1)(l-1)(l+2) W^2 + \mu l(l+1) Z^2 \right] r^2 \, dr \quad \text{(toroidals)} \\ &= \int \left[\frac{\mu}{r^2} l(l+1)(l-1)(l+2) V^2 + \mu l(l+1) X^2 + \frac{\mu}{3} (2U' - F)^2 + K(U' + F)^2 \right] r^2 \, dr \end{aligned} \tag{3.25}$$

for spheroidals. For a transversely isotropic solid :

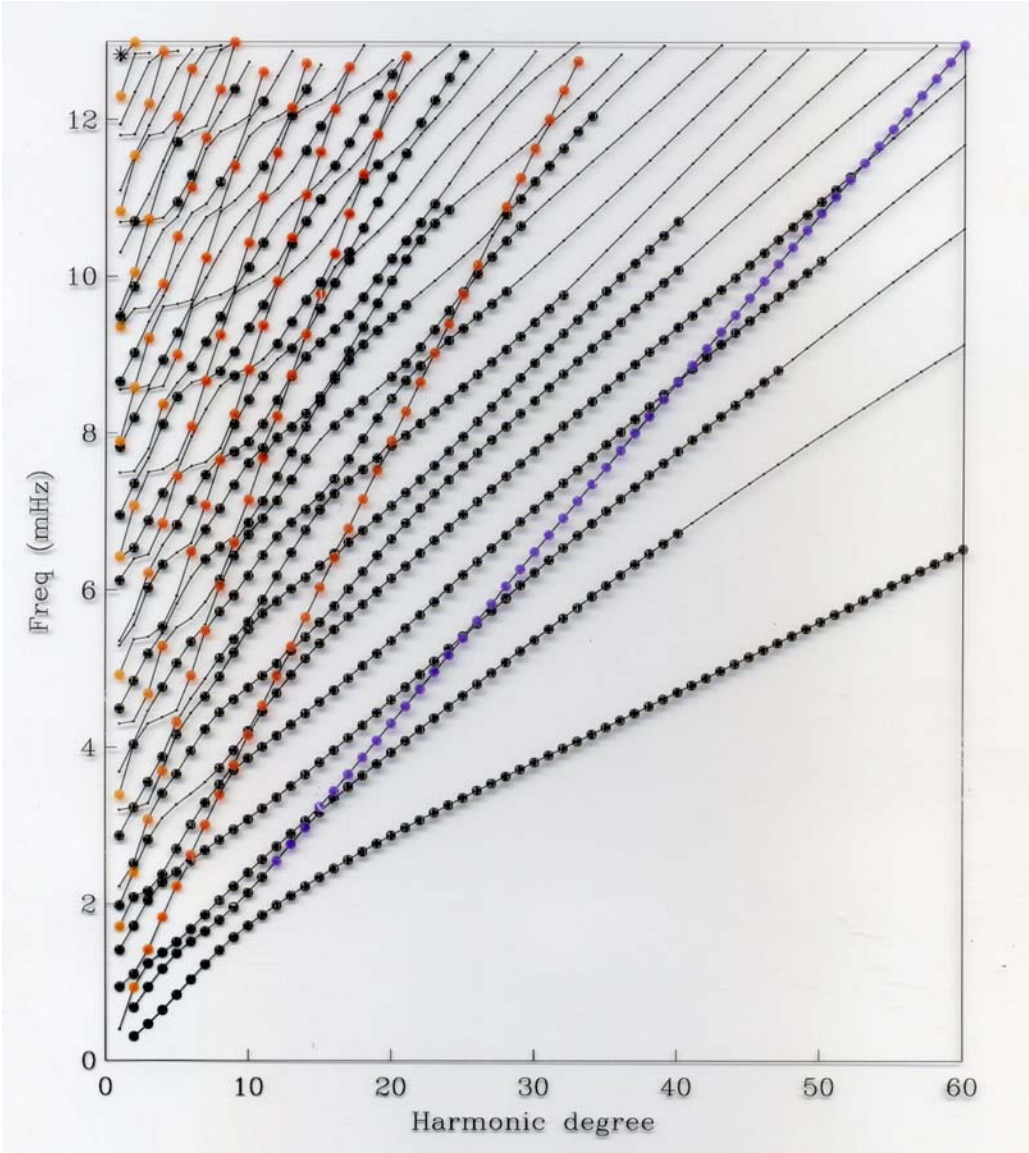
$$\begin{aligned} \int_V \boldsymbol{\epsilon}^* \cdot \cdot \mathbf{C} : \boldsymbol{\epsilon} \, dV &= \int \left[l(l+1)(l-1)(l+2) \frac{\mathrm{N}}{r^2} W^2 + l(l+1) \mathrm{L} Z^2 \right] r^2 \, dr \quad \text{(toroidals)} \\ &= \int \left[l(l+1)(l-1)(l+2) \frac{\mathrm{N}}{r^2} V^2 + l(l+1) \mathrm{L} X^2 + 2\mathrm{F} U' F + (\mathrm{A} - \mathrm{N}) F^2 + \mathrm{C} U'^2 \right] r^2 \, dr \end{aligned} \tag{3.26}$$

for spheroidals.

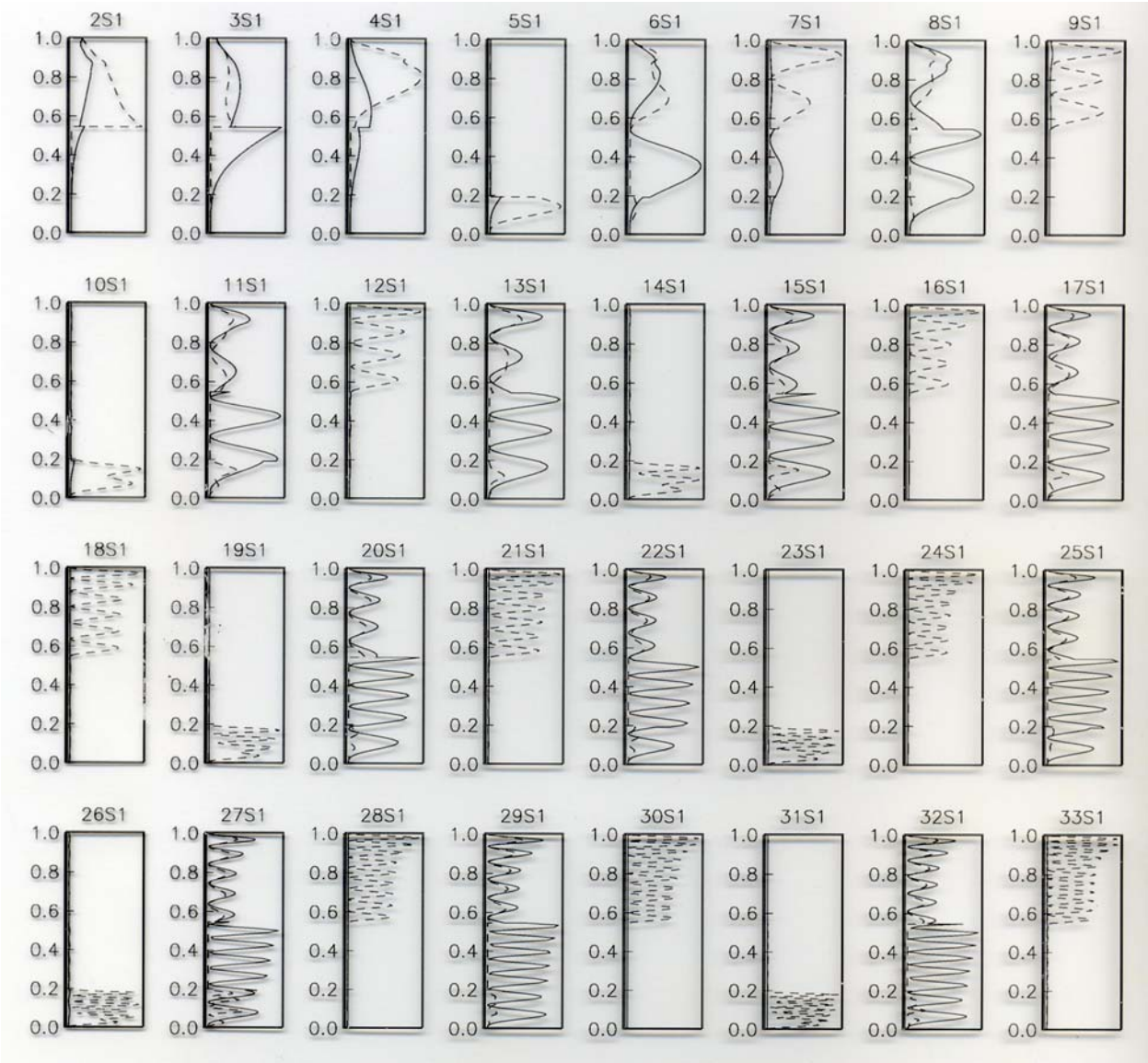


Normalized radius

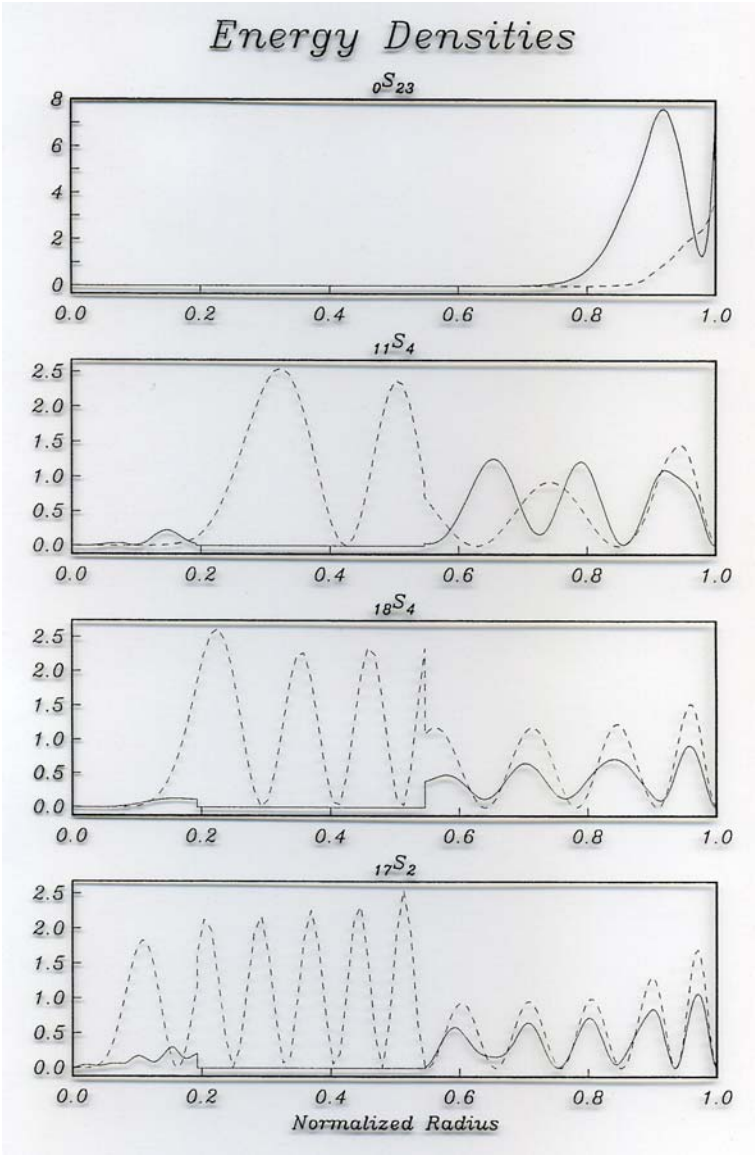
Dash=shear, solid=compressional energy density



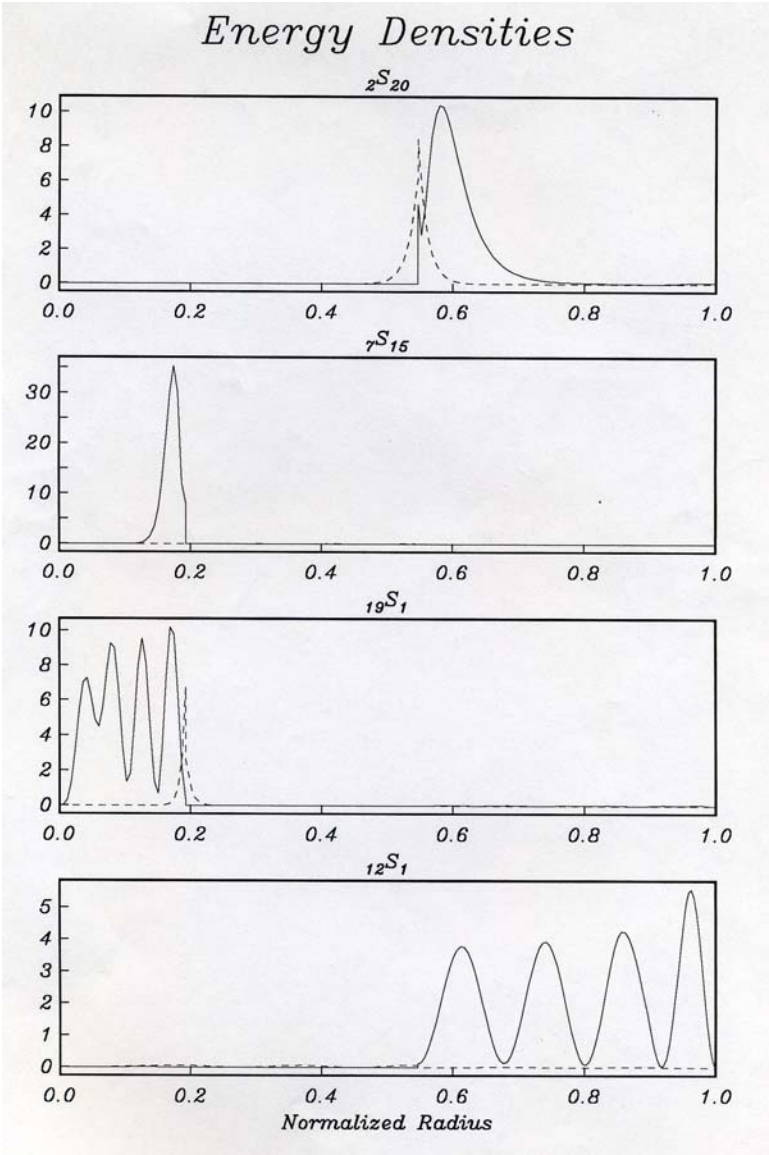
(black dots are observed modes)



All modes for l=1



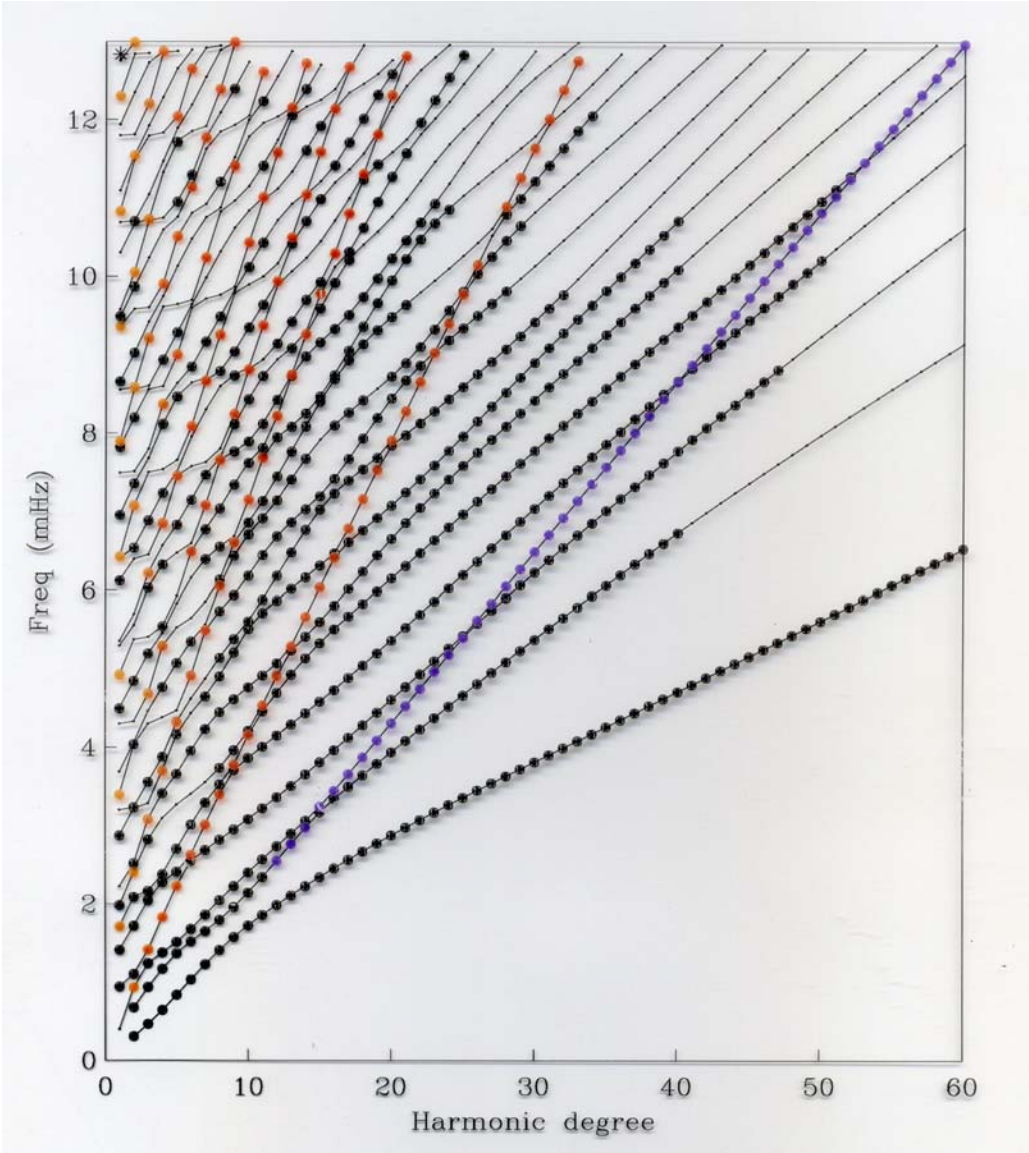
(normal normal modes)



hard to compute

ScS --not observed

(not-so-normal normal modes)


Freq (mHz)
Harmonic degree
0
2
4
6
8
10
12
0
10
20
30
40
50
60

Another problem

- Stoneley and IC modes have part of their eigenfunctions which decay exponentially towards the surface
- As your mother told you, NEVER integrate down an exponential!!
- For these modes, need to do another integration from surface to CMB or ICB to get final eigenfunction (remedy)

Handling attenuation (perturbation theory)

Perturbation theory

We consider the effect of a perturbation in elastic properties and density. Our operator $\mathbf{L}$ becomes perturbed as does the displacement field and frequency, *i.e.*,

$$\begin{aligned}
\mathbf{L} &\rightarrow \mathbf{L} + \delta\mathbf{L} \\
\rho_0 &\rightarrow \rho_0 + \delta\rho_0 \\
\phi_0 &\rightarrow \phi_0 + \delta\phi_0 \\
\mathbf{s}_k &\rightarrow \mathbf{s}_k + \delta\mathbf{s}_k \\
\phi_{1k} &\rightarrow \phi_{1k} + \delta\phi_{1k} \\
\omega_k^2 &\rightarrow \omega_k^2 + \delta\omega_k^2
\end{aligned}$$

and our unperturbed solution satisfies

$$\mathbf{L}(\mathbf{s}_k) + \rho_0\omega_k^2\mathbf{s}_k = 0$$

and we have

$$-\omega_k^2 \int_V \rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k \, dV = \int_V \mathbf{s}_k^\star \mathbf{L}(\mathbf{s}_k) \, dV$$

Perturbing this, and using the properties of the operator $\mathbf{L}$ gives:

$$\delta\omega_k^2 \int_V \rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k \, dV = \int_V \mathbf{s}_k^\star \delta\mathbf{L}(\mathbf{s}_k) \, dV + \omega_k^2 \int_V \delta\rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k \, dV$$

We use this to compute the effects of attenuation

$$\delta\omega_k^2 \int_0^a \rho_0 N r^2\, dr = \int_0^a \left[K'\delta\kappa + M'\delta\mu + R\delta\rho_0\right] r^2\, dr \tag{4.56}$$

where, for spheroidal modes:

$$N = U^2 + l(l+1)V^2$$

$$K' = \left(\frac{dU}{dr} + F\right)^2$$

$$M' = \tfrac{1}{3}\left(2\frac{dU}{dr} - F\right)^2 + \frac{l(l+1)}{r^2}\left(r\frac{dV}{dr} - V + U\right)^2 + \frac{1}{r^2}(l+2)(l-1)l(l+1)V^2$$

$$R = 2U\left(\frac{d\phi_1}{dr} + 4\pi G U \rho_0 - F\frac{d\phi_0}{dr}\right) + \frac{2}{r}V\phi_1 l(l+1) - \int_r^a 8\pi G\rho_0 U F\, dr - \omega_k^2 N$$

and, for toroidal modes:

$$N = l(l+1)W^2$$

$$K' = 0$$

$$M' = \frac{l(l+1)}{r^2}\left(r\frac{dW}{dr} - W\right)^2 + \frac{1}{r^2}(l+2)(l-1)l(l+1)W^2$$

$$R = -\omega_k^2 N$$

Computation of mode attenuation

μ and κ have the form

$$\mu(\omega) = \mu_0(\omega_r)\left[1 + \frac{2}{\pi Q_\mu}\ln\left(\frac{\omega}{\omega_r}\right) + \frac{i}{Q_\mu}\right]$$

where ω_r is a reference frequency. We are currently interested in the small imaginary perturbation, *i.e.*,

$$\mu \to \mu + \delta\mu \quad \text{where} \quad \delta\mu = \frac{i\mu_0}{Q_\mu}$$

and similarly for κ. This results in a perturbation to the squared frequency of the mode, *i.e.*,

$$\omega_k^2 \to \omega_k^2 + \delta\omega_k^2 = \omega_k^2\left[1 + \frac{\delta\omega_k^2}{\omega_k^2}\right]$$

$\delta\omega_k^2$ is purely imaginary, so we write

$$\frac{\delta\omega_k^2}{\omega_k^2} = \frac{i}{Q_k}$$

where Q_k is the Q of the kth mode. We rearrange this to give

$$\delta\omega_k = \frac{i\omega_k}{2Q_k} = i\alpha_k \quad \text{and} \quad \omega_k \to \omega_k + i\alpha_k$$

Beware!

- The attenuation rate of the mode, its group velocity (found by varying harmonic degree) and the kinetic and potential energies are all found by performing numerical integrals using Gauss-Legendre. The mode eigenfunctions are approximated by cubic polynomials between mode knots.

- If you have insufficient knots in your model, these integrals will be imprecise

- Check the output to make sure you are ok

Some final comments

- Many things can go wrong in a mode calculation and this code has been designed to avoid or fix most of them
- You can still break it. For example, if you work at high frequencies and your model has an ocean, you can get Stoneley modes trapped on the ocean floor. The code could be adapted to handle this
- Other versions of the code exist to handle high frequencies -- these may be implemented in CIG eventually
- Other versions have also been designed to read an observed mode list for use in doing 1D reference Earth modeling

A few words about synthetics

Spherical Earth synthetics

Let $\psi_1 = \dot{M}_{rr}, \psi_2 = \dot{M}_{\theta\theta}, \psi_3 = \dot{M}_{\phi\phi}, \psi_4 = \dot{M}_{r\theta}, \psi_5 = \dot{M}_{r\phi}, \psi_6 = \dot{M}_{\theta\phi}$

The mode sum synthetic seismogram is

$$\mathbf{a}(\mathbf{r}, \mathbf{r}_0, t) = \sum_{i=1}^{6} \mathbf{B}_i(\mathbf{r}, \mathbf{r}_0, t) \star \psi_i(t)$$

where the Green's functions are

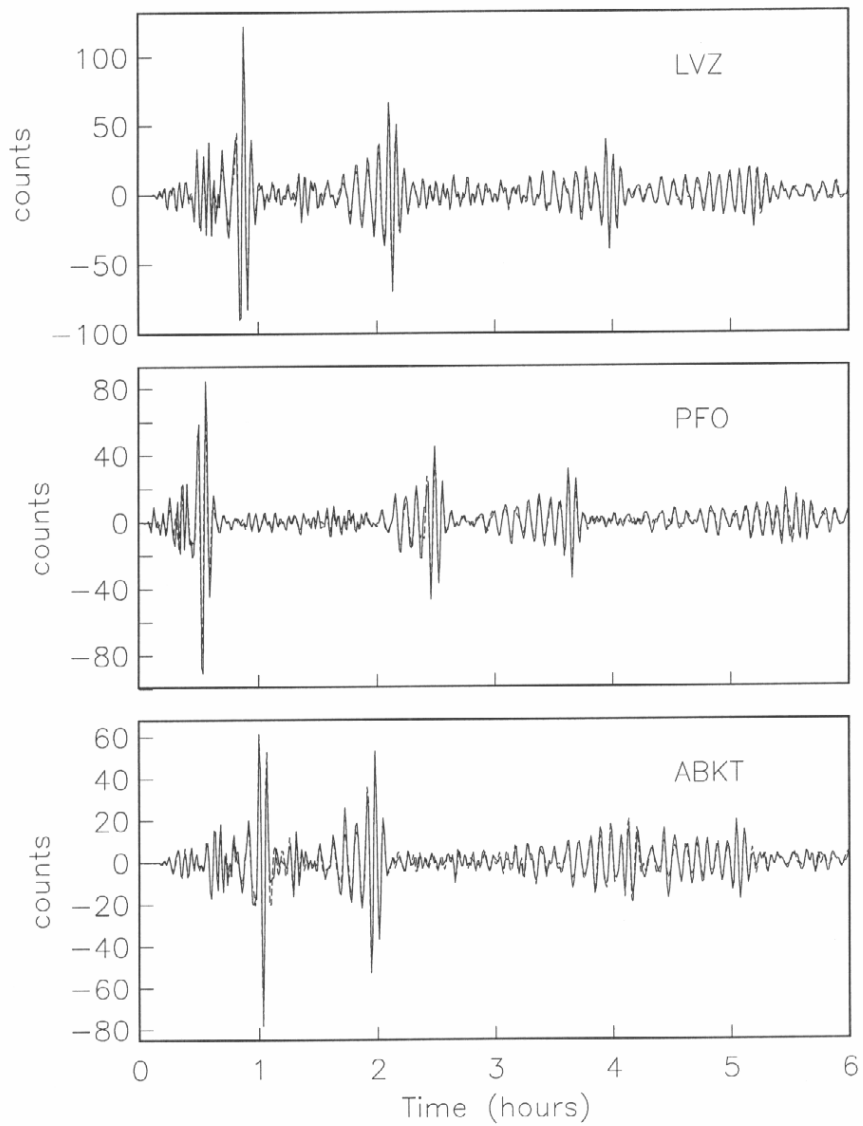
$$\mathbf{B}_i(\mathbf{r}, \mathbf{r}_0, t) = \sum_{n,l} \frac{1}{{}_n\omega_l^2} {}_n\mathbf{G}_l^i(\mathbf{r}, \mathbf{r}_0) {}_nC_l(t)$$

and

$${}_nC_l(t) = [1 - \cos({}_n\omega_l t) e^{-{}_n\alpha_l t}] H(t)$$

If we are given a model of the Earth, we can calculate everything but $\psi_i(t)$ – the source mechanism.

- Program SYNDAT allows you to make synthetics for any general moment tensor or double couple source. A source time function can also be specified but this is assumed to be triangular in shape.
- Finite sources can be modeled by summing point source Green's functions distributed along the rupture
- The equation above can be used to invert for your own moment tensor


LVZ
PFO
ABKT
counts
counts
counts
100
50
0
−50
−100
80
40
0
−40
−80
60
40
20
0
−20
−40
−60
−80
0
1
2
3
4
5
6
Time (hours)

Fluid (isotropic) regions without self-gravitation

Set $d\Phi_1/dr$ and Φ_1 to zero. ($\Psi_1 = 4\pi G\rho_0 U$). Choose

$$\mathbf{y} = \begin{bmatrix} rU \\ rR \end{bmatrix}$$

Then

$$\frac{dy_1}{dr} = -T_{11}y_1 + C_{11}y_2$$
$$\frac{dy_2}{dr} = S_{11}y_1 + T_{11}y_2$$

with

$$T_{11} = -g_0\gamma + \frac{1}{r}$$
$$C_{11} = \frac{1}{\lambda} - \frac{\gamma}{\rho_0}$$
$$S_{11} = -\rho_0\left[\omega_k^2 + \frac{4g_0}{r} - g_0^2\gamma\right] + 4\pi G\rho_0^2$$

where $\mathcal{L} = \sqrt{l(l+1)}$ and $\gamma = \mathcal{L}^2/(r^2\omega_k^2)$
Note that y_4, y_5, and y_6 at the free surface.

Numerical considerations – scalings

We are dealing with mixed dimensions so it makes sense to non-dimensionalize everything. A natural set of scales is

- length $\rightarrow$ radius of the Earth, a
- density $\rightarrow$ mean density of the Earth, $\bar{\rho}$
- time $\rightarrow 1/\sqrt{\pi G \bar{\rho}}$ where G is Newton's constant.

This time unit is about 930 seconds which is reasonable for long-period calculation. Note that $g_0(a) = 4/3$ with these units.

We should also take care that $\mathbf{y}$ is chosen so that all the elements of the matrix $\mathbf{A}$ are similar in magnitude. This accounts for the $\sqrt{l(l+1)}$ scaling in y_2 and y_5.